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Théorie non maxwellienne des plasmas homogènes et anisotropes.

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Résumé. — En s'appuyant sur une méthode de résolution de l'équation intégro-différentielle de Boltzmann établie par les auteurs ([11]) qui généralise des résultats antérieurs ([4]), on calcule la fonction de répartition des vitesses électroniques dans un gas ionisé anisotrope (soumis à un champ magnétique constant) pour des états *non-maxwelliens*. Celle-ci permet d'obtenir des expressions explicites pour la conductivité magnéto-ionique, pour le tenseur diélectrique, pour l'effet Hall, pour la deviation d'un faisceau électronique et pour une généralisation de la formule de mobilité de Langevin. On procède à une comparaison de ces résultats avec ceux fournis par d'autres méthodes de calcul (basées sur le libre parcours moyen). On étudie ensuite la propagation des ondes électromagnétiques planes dans un tel plasma; en particulier, on établit des formules concernant l'indice de réfraction, la biréfringence, la vitesse de phase et de groupe, l'affaiblissement, la polarisation des ondes et les frequences limites. Moyennant certaines approximations, dont la validité est discutée, on retrouve les résultats des théories classiques relatives à l'ionosphère (APPLETON, HARTREE, etc.).

L'objet du présent mémoire est la théorie magnéto-ionique de milieux gazeux légèrement ionisés, fondée non pas sur des méthodes approchées s'appuyant sur la notion du libre parcours moyen, mais sur l'équation intégro-différentielle de Boltzmann.

Pour préciser les idées, rappelons brièvement certains caractères d'un gaz idéal composé d'une seule espèce de constituants, en particulier la notion de la distribution des vitesses moléculaires $\boldsymbol{v}$ (de composantes v_x, v_y, v_z).

Soit :

$$f(\boldsymbol{v})\,\mathrm{d}v_x\,\mathrm{d}v_y\,\mathrm{d}v_z = f(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}$$

le nombre de particules par unité de volume dont les composantes de vitesse sont comprises entre v_x et $v_x+\mathrm{d}v_x$, v_y et $v_y+\mathrm{d}v_y$, v_z et $v_z+\mathrm{d}v_z$; f est par definition la fonction de répartition des vitesses $\boldsymbol{v}$.

L'integrale triple de cette fonction étendue aux composantes de vitesse

$$n = \int_0^\infty\int_0^\infty\int_0^\infty f(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}\,,$$

est la densité des molécules. Soit G une fonction quelconque de la vitesse corpusculaire; la valeur moyenne ou l'espérance mathématique $E(G)$ de G se définit par l'expression :

$$\bar{G} = E(G) = \frac{1}{n}\iiint Gf\,\mathrm{d}\boldsymbol{v}\,.$$

Si le gaz est uniforme et non soumis à un champ de forces extérieur, la fonction f représente la distribution maxwellienne dans les conditions classiques :

$$f = n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left[-\frac{mv^2}{2kT}\right], \qquad (v = |\boldsymbol{v}|)\,.$$

La répartition des vitesses des particules chargées dans un milieu gazeux ayant une température et une pression uniformes n'est pas nécessairement maxwellienne si leur densité n'est pas uniforme ou si des champs extérieurs sont mis en jeu. Dans de tels cas, la fonction de distribution des vitesses f ne dépendra pas seulement de la vitesse $\boldsymbol{v}$, mais aussi de la position $\boldsymbol{r}$ et du temps t. L'évolution de la fonction $f(\boldsymbol{v}, \boldsymbol{r}, t)$ sera des lors régie par l'équation intégro-différentielle de Boltzmam [1-3].

1. – Équations générales.

On considère le gaz ionisé comme un mélange de molécules neutres et d'électrons et l'on suppose ce milieux soumis à l'action d'un champ électrique oscillant $\boldsymbol{E}\cos\omega t$ et d'un champ magnétique $\boldsymbol{H}_0$; on suppose de plus que les chocs entre électrons et molécules sont élastiques. (Nous ne traitons pas le cas où des ions seraient présents, car la contribution ionique à la conductivité électrique est généralement négligeable; d'autre part il suffirait, pour tenir

(1) L. Boltzman : *Vorlesungen über Gastheorie* (1895).

(2) D. Hilbert : *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (1912).

(3) S. Chapman et T. Cowling : *The mathematical theory of non uniform gases* (1939).

compte de cette contribution, d'appliquer les résultats usuels de la théorie maxwellienne que nous voulons ici généraliser).

Pour déterminer les propriétés électriques de ce milieu, il est nécessaire de connaître les fonctions de répartition des vitesses pour les constituants du gaz: on fera ce calcul pour l'état stationnaire en utilisant l'équation intégro-différentielle de Maxwell-Boltzmann.

Si f_1 et f_2 sont les fonctions de répartition respectives des molécules et des électrons, on a les deux équations de Boltzmann:

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v}_1 \cdot \mathbf{grad}_{r_1} f_1 = \frac{\partial e f_1}{\partial t}, \tag{1}$$

$$\frac{\partial f_2}{\partial t} + \boldsymbol{v}_2 \cdot \mathbf{grad}_{r_2} f_2 + \boldsymbol{F}_2 \cdot \mathbf{grad}_{v_2} f_2 = \frac{\partial e f_2}{\partial t}, \tag{2}$$

(où $\boldsymbol{v}_1$ et $\boldsymbol{v}_2$ désignent respectivement la vitesse des molécules et des électrons). Dans l'équation (1) on suppose qu'aucune force extérieure n'agit sur les molécules; d'autre part $\boldsymbol{F}_2$ représente l'action électromagnétique sur les électrons du champ $(\boldsymbol{E}, \boldsymbol{H}_0)$; enfin $\partial e f_1/\partial t$ et $\partial e f_2/\partial t$ sont les contributions des chocs entre molécules et électrons.

Si l'on s'intéresse à l'état ou l'on a $\mathbf{grad}_{r_1} f_1 = \mathbf{grad}_{r_2} f_2 = 0$, f_1 et f_2 sont seulement fonctions de $\boldsymbol{v}_1$ et $\boldsymbol{v}_2$ et du temps éventuellement. Nous supposerons de plus dans la suite des calculs que le gaz ionisé peut être traité comme un gaz de Lorentz; ce dernier répond aux deux conditions suivantes;

a) la masse m_1 des molécules du 1er constituant est grande par rapport à la masse des particules du 2^e constituant;

b) l'influence des chocs mutuels entre les particules du 2^e constituant est négligeable comparée à celle de leurs chocs avec les molécules du 1er constituant.

On voit que la condition *a*) est réalisée sans hypothèse supplementaire, étant donné la petitesse du rapport m_2/m_1 (m_1 et m_2 désignant respectivement la masse des molécules et des électrons); par contre, la condition *b*) n'est réalisée que si la densité électronique est faible; les calculs qui suivent ne sont donc valables que pour les gaz faiblement ionisés.

Cette hypothèse étant admise, on en tire deux conséquences:

1) Dans les chocs électrons-molécules, la variation de vitesse des molécules sera négligeable en moyenne, de sorte que la fonction f_1 est peu affectée par les chocs de cette nature; donc $f_1(v_1)$ est maxwellien dans l'état stationnaire et l'on peut remplacer (1) par

$$f_1(v_1) = n_1 \left(\frac{m_1}{2\pi k T}\right)^{\frac{3}{2}} \exp\left[-\frac{m_1 v_1^2}{2kT}\right], \tag{3}$$

ou n_1 est le nombre de molécules par unité de volume et T la température absolue du gaz.

2) Dans l'équation (2), la seule contribution appréciable du terme $\partial e f_2/\partial t$ sera fournie par les chocs molécules-électrons. Le problème est donc ramené à l'étude de l'équation (2) dans cette hypothèse. On a:

$$(4) \qquad \boldsymbol{F}_2 = \frac{e_2}{m_2}\boldsymbol{E}\cos\omega t + \frac{e_2}{m_2}(\boldsymbol{v}_2 \wedge \boldsymbol{H}_0) = \boldsymbol{\Gamma}_2 \cos\omega t + \frac{e_2}{m_2}(\boldsymbol{v}_2 \wedge \boldsymbol{H}_0)\,,$$

où e_2 désigne la charge de l'électron. En explicitant $\partial e f_2/\partial t$, l'équation (2) s'écrit alors:

$$(5) \qquad \frac{\partial f_2}{\partial t} + \left[\boldsymbol{\Gamma}_2 \cos\omega t + \frac{e_2}{m_2}(\boldsymbol{v}_2 \wedge \boldsymbol{H}_0)\right]\cdot \mathbf{grad}_{v_2} f_2 =$$

$$= \iiint\!\!\iint\!\!\iint (f_1' f_2' - f_1 f_2) g b \,\mathrm{d}b \,\mathrm{d}\varepsilon \,\mathrm{d}\boldsymbol{v}_1\,,$$

f_1' et f_2' sont les fonctions f_1 et f_2 prises pour les valeurs v_1' et v_2'; v_1' et v_2' étant les vitesses respectives des particules après un choc où les vitesses initiales étaient $\boldsymbol{v}_1$ et $\boldsymbol{v}_2$; g est la vitesse relative des deux particules ($g = |\boldsymbol{v}_1 - \boldsymbol{v}_2| = |\boldsymbol{v}_1' - \boldsymbol{v}_2'|$) et h et ε sont les paramètres d'impact.

2. – Calcul de l'approximation d'ordre zéro.

La présence de la force de Lorentz dans (5) entraine que f_2 n'est pas maxwellien, l'écart avec l'état maxwellien étant d'autant plus sensible que les champs $\boldsymbol{E}$ et $\boldsymbol{H}_0$ sont plus intenses. Nous nous intéressons d'emblée aux cas où ces champs peuvent être d'intensité quelconque et nous utilisons pour cela une méthode d'approximations successives mettant en évidence les termes d'anisotropie en $\boldsymbol{\Gamma}_2$, $\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2$, $\boldsymbol{H}_0 \wedge (\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)$. Développons donc f_2 en fonction de ses harmoniques sphériques, soit (en se limitant à la 1ère approximation de la fonction de distribution):

$$(6) \qquad f_2 = f_2^{(0)} + (\boldsymbol{\Gamma}_2 \boldsymbol{v}_2)(\alpha_2^{(1)} \cos\omega t + \beta_2^{(1)} \sin\omega t) +$$

$$+ (\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)\boldsymbol{v}_2(\xi_2^{(1)} \cos\omega t + \eta_2^{(1)} \operatorname{sen}\omega t) +$$

$$+ [\boldsymbol{H}_0 \wedge (\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)]\boldsymbol{v}_2(\gamma_2^{(1)} \cos\omega t + \delta_2^{(1)} \sin\omega t)\,,$$

où $f_2^{(0)}$, $\alpha_2^{(1)}$, $\beta_2^{(1)}$, $\xi_2^{(1)}$, $\eta_2^{(1)}$, $\gamma_2^{(1)}$ et $\delta_2^{(1)}$ ne dépendent que de v_2 ($v_2 = |\boldsymbol{v}_2|$).

Pour résoudre (5), il nous faut calculer les 7 fonctions précédentes; pour cela nous portons (6) dans (5) et identifions les deux membres. Nous verrons

que $f_2^{(0)}$, dans le cas général où $\mathbf{\Gamma}_2$ est quelconque, n'est pas maxwellien et nous montrerons que la théorie maxwellienne est un cas limite de la théorie générale pour certaines conditions.

Portons donc (6) dans (5); on doit calculer des termes de la forme suivant

$$\mathbf{grad}_{v_2} f_2^{(0)} = \frac{\boldsymbol{v}_2}{v_2}\frac{\partial f_2^{(0)}}{\partial v_2}, \quad \mathbf{grad}\,[\alpha_2^{(1)}(\mathbf{\Gamma}_2\boldsymbol{v}_2)] = \alpha_2^{(1)}\mathbf{\Gamma}_2 + (\mathbf{\Gamma}_2\cdot\boldsymbol{v}_2)\frac{\boldsymbol{v}_2}{v_2}\frac{\partial\alpha_2^{(1)}}{\partial v_2},$$

et un terme analogue en $\beta_2^{(1)}$

$$\mathbf{grad}_{v_2}[(\boldsymbol{H}_0\wedge\mathbf{\Gamma}_2)\boldsymbol{v}_2\xi_2^{(1)}] = (\boldsymbol{H}_0\wedge\mathbf{\Gamma}_2)\xi_2^{(1)} + (\boldsymbol{H}_0, \mathbf{\Gamma}_2, \boldsymbol{v}_2)\frac{\boldsymbol{v}_2}{v_2}\frac{\partial\xi_2^{(1)}}{\partial v_2},$$

$$\mathbf{grad}_{v_2}\{[\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\mathbf{\Gamma}_2)]\boldsymbol{v}_2\gamma_2^{(1)}\} = [\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\mathbf{\Gamma}_2)]\gamma_2^{(1)} + \{[\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\mathbf{\Gamma}_2)]\cdot\boldsymbol{v}_2\}\frac{\boldsymbol{v}_2}{v_2}\frac{\partial\gamma_2^{(1)}}{\partial v_2},$$

et des termes analogues respectivement en $\eta_2^{(1)}$ et $\delta_2^{(1)}$.

Nous devons maintenant effectuer les produits $\mathbf{\Gamma}_2\,\mathbf{grad}_{v_2} f_2$, ce qui donne des expressions de la forme:

$$(7)\quad \begin{cases} \dfrac{\mathbf{\Gamma}_2\cdot\boldsymbol{v}_2}{v_2}\dfrac{\partial f_2^{(0)}}{\partial v_2};\ \alpha_2^{(1)}\Gamma_2^2 + (\mathbf{\Gamma}_2\cdot\boldsymbol{v}_2)\dfrac{\mathbf{\Gamma}_2\cdot\boldsymbol{v}_2}{2}\dfrac{\partial\alpha_2^{(1)}}{\partial v_2}(+\text{ terme en }\beta_2^{(1)}), \\ (\boldsymbol{H}_0, \mathbf{\Gamma}_2, \boldsymbol{v}_2)\dfrac{\mathbf{\Gamma}_2\cdot\boldsymbol{v}_2}{v_2}\dfrac{\partial\xi_2^{(1)}}{\partial v_2}(+\text{ terme en }\eta_2^{(1)}), \\ \mathbf{\Gamma}_2[\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\mathbf{\Gamma}_2)]\gamma_2^{(1)} + \{[\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\mathbf{\Gamma}_2)]\boldsymbol{v}_2\}\dfrac{\mathbf{\Gamma}_2\cdot\boldsymbol{v}_2}{v_2}\dfrac{\partial\gamma_2^{(1)}}{\partial v_2}(+\text{ terme en }\delta_2^{(1)}). \end{cases}$$

Les produits $(\boldsymbol{v}_2\wedge\boldsymbol{H}_0)\,\mathbf{grad}_{v_2} f_2$ nous donnent aussi:

$$(8)\quad \begin{cases} (\boldsymbol{v}_2, \boldsymbol{H}_0, \mathbf{\Gamma}_2)\alpha_2^{(1)}\ (\text{et terme en }\beta_2^{(1)}), \\ (\boldsymbol{v}_2\wedge\boldsymbol{H}_0)(\boldsymbol{H}_0\wedge\mathbf{\Gamma}_2)\xi_2^{(1)} = [(\boldsymbol{H}_0\cdot\mathbf{\Gamma}_2)(\boldsymbol{H}_0\cdot\boldsymbol{v}_2) - H_0^2(\mathbf{\Gamma}_2\cdot\boldsymbol{v}_2)]\xi_2^{(1)}\ (\text{et terme en }\eta_2^{(1)}), \\ (\boldsymbol{v}_2\wedge\boldsymbol{H}_0)[\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\mathbf{\Gamma}_2)]\gamma_2^{(1)} = [-H_0^2(\boldsymbol{v}_2, \boldsymbol{H}_0, \mathbf{\Gamma}_2)\gamma_2^{(1)}]\ (\text{et terme en }\delta_2^{(1)}). \end{cases}$$

Afin de procéder à l'identification des deux membres de (5) remarquons que l'intégrale du second membre comprend un terme scalaire:

$$(9)\qquad \iiiint (f_1' f_2'^{(0)} - f_1 f_2^{(0)}) g b\,\mathrm{d}b\cdot\mathrm{d}\varepsilon\,\mathrm{d}\boldsymbol{v}_1,$$

un terme en $(\mathbf{\Gamma}_2\cdot\boldsymbol{v}_2)$:

$$(10)\qquad \iiiint \{f_1'(\alpha_2^{(1)}\cos\omega t + \beta_2^{(1)}\sin\omega t)(\mathbf{\Gamma}_2\cdot\boldsymbol{v}_2') - f_1(\alpha_2^{(1)}\cos\omega t + \beta_2^{(1)}\sin\omega t)(\mathbf{\Gamma}_2\boldsymbol{v}_2)\}g b\,\mathrm{d}b\,\mathrm{d}\varepsilon\,\mathrm{d}\boldsymbol{v}_1,$$

un terme en $(\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)\boldsymbol{v}_2$:

$$(11) \qquad \iiiiint \{f_1'(\xi_2^{(1)} \cos \omega t + \eta_2^{(1)} \sin \omega t)[(\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)\boldsymbol{v}_2'] - f_1(\xi_2^{(1)} \cos \omega t + \eta_2^{(1)} \sin \omega t)[(\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)\boldsymbol{v}_2]\} g b \, \mathrm{d}b \, \mathrm{d}\varepsilon \, \mathrm{d}\boldsymbol{v}_1 ,$$

et un terme en $\{[\boldsymbol{H}_0 \wedge (\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)]\boldsymbol{v}_2\}$:

$$(12) \qquad \iiiiint \{f_1'(\gamma_2^{(1)} \cos \omega t + \delta_2^{(1)} \sin \omega t) \cdot [\boldsymbol{H}_0 \wedge (\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)\boldsymbol{v}_2'] - f_1(\gamma_2^{(1)} \cos \omega t + \delta_2^{(1)} \sin \omega t)[\boldsymbol{H}_0 \wedge (\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)\boldsymbol{v}_2]\} g b \, \mathrm{d}b \, \mathrm{d}\varepsilon \, \mathrm{d}\boldsymbol{v}_1 .$$

Les intégrales (10), (11) et (12) s'évaluent aisément en remarquant que, d'après les conditions qui définissent un gaz de Lorentz, on peut poser $v_2' = v_2 = g$ et se servir de l'équation (3). Nous obtenons alors, en introduisant le libre parcours moyen d'un électron de vitesse v_2, $\lambda(v_2)$:

pour (10):

$$(10') \qquad -\frac{(\alpha_2^{(1)} \cos \omega t + \beta_2^{(1)} \sin \omega t)}{\lambda} v_2 (\boldsymbol{\Gamma}_2 \cdot \boldsymbol{v}_2) ,$$

pour (11):

$$(11') \qquad -\frac{(\xi_2^{(1)} \cos \omega t + \eta_2^{(1)} \sin \omega t)}{\lambda} v_2 [(\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)\boldsymbol{v}_2] ,$$

pour (12):

$$(12') \qquad -\frac{(\gamma_2^{(1)} \cos \omega t + \delta_2^{(1)} \sin \omega t)}{\lambda} v_2 \{[\boldsymbol{H}_0 \wedge (\boldsymbol{H}_0 \wedge \boldsymbol{\Gamma}_2)]\boldsymbol{v}_2\} .$$

(12′) se décompose en 2 termes, l'un en $(\boldsymbol{H}_0 \cdot \boldsymbol{v}_2)$ et l'autre en $\boldsymbol{\Gamma}_2 \cdot \boldsymbol{v}_2$ selon la formule:

$$(13') \qquad -\frac{(\gamma_2^{(1)} \cos \omega t + \delta_2^{(1)} \sin \omega t) v_2}{\lambda} \{(\boldsymbol{H}_0 \cdot \boldsymbol{\Gamma}_2)(\boldsymbol{H}_0 \cdot \boldsymbol{v}_2) - H_0^2 (\boldsymbol{\Gamma}_2 \cdot \boldsymbol{v}_2)\} .$$

Les expressions (10′), (11′) et (13′) nous montrent que nous devons identifier dans (5) les termes en $(\boldsymbol{\Gamma}_2 \cdot \boldsymbol{v}_2)$, $(\boldsymbol{H}_0, \boldsymbol{\Gamma}_2, \boldsymbol{v}_2)$ et $(\boldsymbol{H}_0 \cdot \boldsymbol{v}_2)$ respectivement. Comme dans (10′), (11′) et (13′) ces termes ne dépendent que de v_2 et non de sa direction, nous devrons prendre les moyennes sur les directions de v_2 pour les expressions analogues de (7) et (8) qui dépendent de cette direction. On

a ainsi:

$$(14)\quad \begin{cases} \overline{\dfrac{(\Gamma_2\cdot v_2)\,(\boldsymbol{\Gamma}_2\cdot\boldsymbol{v}_2)}{v_2}}\,\dfrac{\partial\alpha_2^{(1)}}{\partial v_2} = \dfrac{\Gamma_2^2 v_2}{3}\,\dfrac{\partial\alpha_2^{(1)}}{\partial v_2}\ (+\text{ terme en }\beta_2^{(1)}), \\ \overline{[(\boldsymbol{H}_0\wedge\boldsymbol{\Gamma}_2)\boldsymbol{v}_2]\,\dfrac{(\Gamma_2 v_2)}{v_2}}\,\dfrac{\partial\xi_2^{(1)}}{\partial v_2} = 0\ , \\ \overline{\{[\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\boldsymbol{\Gamma}_2)]\boldsymbol{v}_2\}\,\dfrac{(\boldsymbol{\Gamma}_2\cdot\boldsymbol{v}_2)}{v_2}}\,\dfrac{\partial\gamma_2^{(1)}}{\partial v_2} = -H_0^2\Gamma_2^2\sin^2\varphi\,\dfrac{v_2}{3}\,\dfrac{\partial\gamma_2^{(1)}}{\partial v_2}\ (+\text{ terme en }\delta_2^{(1)}). \end{cases}$$

(Nous avons par exemple:

$$\overline{\{[\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\boldsymbol{\Gamma}_2)]\boldsymbol{v}_2\}(\boldsymbol{\Gamma}_2\cdot\boldsymbol{v}_2)} = \overline{[(\boldsymbol{H}_0\cdot\boldsymbol{\Gamma}_2)(\boldsymbol{H}_0\cdot\boldsymbol{v}_2) - H_0^2(\boldsymbol{\Gamma}_2\cdot\boldsymbol{v}_2)](\boldsymbol{\Gamma}_2\cdot\boldsymbol{v}_2)} =$$

$$= [H_0\Gamma_2\cos\varphi\,\overline{(H_x v_x + H_y v_y + H_z v_z)(\Gamma_x v_x + \Gamma_y v_y + \Gamma_z v_z)(-H_0^2\overline{(\Gamma_x v_x + \Gamma_y v_y + \Gamma_z v_z)^2}}] =$$

$$= \left[H_0\Gamma_2\cos\varphi(H_x\Gamma_x + H_y\Gamma_y + H_z\Gamma_z)\,\frac{v_2^2}{3} - H_0^2\Gamma_2^2\,\frac{v_2^2}{3}\right] =$$

$$= (H_0^2\Gamma_2^2\cos^2\varphi - H_0^2\Gamma_2^2)\,\frac{v_2^2}{3} = -H_0^2\Gamma_2^2\sin^2\varphi\cdot\frac{v_2^2}{3}\Big)\ ,$$

φ est l'angle entre $\boldsymbol{E}$ et $\boldsymbol{H}_0$.

Nous sommes maintenant en mesure d'identifier les deux membres de (5) (le calcul de $\partial f_2/\partial t$ étant immédiat). Il vient pour le terme scalaire auquel (14) fournit une contribution supplémentaire:

$$(15)\quad \Gamma_2^2\cos\omega t\left\{(\alpha_2^{(1)}\cos\omega t + \beta_2^{(1)}\sin\omega t) + \frac{v_2}{3}\left(\frac{\partial\alpha_2^{(1)}}{\partial v_2}\cos\omega t + \frac{\partial\beta_2^{(1)}}{\partial v_2}\sin\omega t\right) - \right.$$

$$\left. - H_0^2\sin^2\varphi\left[(\gamma_2^{(1)}\cos\omega t + \delta_2^{(1)}\sin\omega t) + \frac{v_2}{3}\left(\frac{\partial\gamma_2^{(1)}}{\partial v_2}\cos\omega t + \frac{\partial\delta_2^{(1)}}{\partial v_2}\sin\omega t\right)\right]\right\} =$$

$$= \iiint\!\!\iiint (f_1' f_2^{(0)\prime} - f_1 f_2^{(0)})\,g b\,\mathrm{d}b\,\mathrm{d}\varepsilon\,\mathrm{d}\boldsymbol{v}_1\ .$$

Les termes en $(\boldsymbol{\Gamma}_2\cdot\boldsymbol{v}_2)$, $(\boldsymbol{H}_0\wedge\boldsymbol{\Gamma}_2)\boldsymbol{v}_2$ et $(\boldsymbol{H}_0\cdot\boldsymbol{v}_2)$ donnent d'autre part respectivement:

$$(16)\quad \frac{\partial f_2^{(0)}}{\partial v_2}\cos\omega t + \omega v_2(-\alpha_2^{(1)}\sin\omega t + \beta_2^{(1)}\cos\omega t) -$$

$$- \frac{e_2 H_0^2}{m_2}\,v_2(\xi_2^{(1)}\cos\omega t + \eta_2^{(1)}\sin\omega t) - \omega v_2 H_0^2\cdot(-\gamma_2^{(1)}\sin\omega t + \delta_2^{(1)}\cos\omega t) =$$

$$= -\frac{(\alpha_2^{(1)}\cos\omega t + \beta_2^{(1)}\sin\omega t)v_2^2}{\lambda} + \frac{H_0^2(\gamma_2^{(1)}\cos\omega t + \delta_2^{(1)}\sin\omega t)v_2^2}{\lambda}\ ;$$

$$(17)\qquad \omega(-\xi_2^{(1)}\sin\omega t+\eta^{(1)}\cos\omega t)+\frac{e_2}{m_2}(\alpha_2^{(1)}\cos\omega t+\beta_2^{(1)}\sin\omega t)-$$
$$-\frac{e_2}{m_2}H_0^2(\gamma_2^{(1)}\cos\omega t+\delta_2^{(1)}\sin\omega t)=-\frac{(\xi_2^{(1)}\cos\omega t+\eta_2^{(1)}\sin\omega t)v_2}{\lambda};$$

$$(18)\qquad \omega(-\gamma_2^{(1)}\sin\omega t+\delta_2^{(1)}\cos\omega t)+$$
$$+\frac{e_2}{m_2}(\xi_2^{(1)}\cos\omega t+\eta_2^{(1)}\sin\omega t)=-\frac{(\gamma_2^{(1)}\cos\omega t+\delta_2^{(1)}\sin\omega t)v_2}{\lambda}.$$

Pour résoudre (15) multiplions les 2 membres par $d\boldsymbol{v}_2=4\pi v_2^2\,dv_2$ et intégrons de 0 a $\boldsymbol{v}_2$; il vient:

$$\tfrac{4}{3}\pi\Gamma_2^2v_2^3\cos\omega t[(\alpha_2^{(1)}\cos\omega t+\beta_2^{(1)}\sin\omega t)-H_0^2\sin^2\varphi(\gamma_2^{(1)}\cos\omega t+\delta_2^{(1)}\sin\omega t)]=$$
$$=\iiiint\!\!\iiiint(f_1'f_2^{(0)\prime}-f_1f_2^{(0)})gb\,db\,d\varepsilon\,d\boldsymbol{v}_1\,d\boldsymbol{v}_2.$$

Le calcul de l'intégrale du second membre s'effectue comme l'indiquent Chapman et Cowling ([3]) et l'on trouve

$$(19)\qquad \tfrac{1}{3}\Gamma_2^2\cos\omega t\,[\alpha_2^{(1)}\cos\omega t+\beta_2^{(1)}\sin\omega t)-H_0^2\sin^2\varphi(\gamma_2^{(1)}\cos\omega t+\delta_2^{(1)}\sin\omega t)]=$$
$$=\frac{kT}{m_1\lambda}\frac{\partial f_2^{(0)}}{\partial v_2}+\frac{m_2v_2}{m_1\lambda}f_2^{(0)}.$$

Le 2ème membre de (19) ne dépendant pas du temps, nous devons prendre la valeur moyenne du 1er sur une période; l'équation (19) doit donc être remplacée par:

$$(19')\qquad \frac{\Gamma_2^2}{6}(\alpha_2^{(1)}-H_0^2\sin^2\varphi\gamma_2^{(1)})=\frac{kT}{m_1\lambda}\frac{\partial f_2^{(0)}}{\partial v_2}+\frac{m_2v_2}{m_1\lambda}f_2^{(0)},$$

d'autre part, les équations (16), (17) et (18) donnent chacune 2 équations par identifications des termes en $\sin\omega t$ et $\cos\omega t$; soit:

$$(16')\qquad \frac{\partial f_2^{(0)}}{\partial v_2}+\omega v_2\beta_2^{(1)}-\frac{e_2H_0^2}{m_2}v_2\xi_2^{(1)}-\omega v_2H_0^2\delta_2^{(1)}=-\frac{\alpha_2^{(1)}v_2^2}{\lambda}+\frac{H_0^2v_2^2\gamma_2^{(1)}}{\lambda};$$

$$(16'')\qquad -\omega v_2\alpha_2^{(1)}-\frac{e_2H_0^2}{m_2}v_2\eta_2^{(1)}+\omega v_2H_0^2\gamma_2^{(1)}=-\frac{\beta_2^{(1)}v_2^2}{\lambda}+\frac{H_0^2v_2^2\delta_2^{(1)}}{\lambda};$$

$$(17')\qquad \omega\eta_2^{(1)}+\frac{e_2}{m_2}\alpha_2^{(1)}-\frac{e_2}{m_2}H_0^2\gamma_2^{(1)}=-\frac{\xi_2^{(1)}v_2}{\lambda};$$

$$(17'')\qquad -\omega\xi_2^{(1)}+\frac{e_2}{m_2}\beta_2^{(1)}-\frac{e_2}{m_2}H_0^2\delta_2^{(1)}=-\frac{\eta_2^{(1)}v_2}{\lambda};$$

$$\omega\delta_2^{(1)} + \frac{e_2}{m_2}\xi_2^{(1)} = -\frac{\gamma_2^{(1)} v_2}{\lambda}\,; \tag{18'}$$

$$-\omega\gamma_2^{(1)} + \frac{e_2}{m_2}\eta_2^{(1)} = -\frac{\delta_2^{(1)} v_2}{\lambda}\,. \tag{18''}$$

Les 7 équations (19'), (16'), (16''), (17'), (17''), (18') et (18'') permettent de calculer les 7 fonctions $f_2^{(0)}$, $\alpha_2^{(1)}$, $\beta_2^{(1)}$, $\xi_2^{(1)}$, $\eta_2^{(1)}$, $\gamma_2^{(1)}$ et $\delta_2^{(1)}$. Si l'on pose:

$$z(v_2) = \frac{e_2^2 H_0^2 \lambda^2}{m_2^2(\lambda^2\omega^2 + v_2^2)}\,; \tag{20}$$

$$A(v_2) = \frac{\lambda^2\omega^2(1-z)^2 + v_2^2(1+z)^2}{\lambda(1+z)}\,; \tag{21}$$

$$B(v_2) = \frac{\lambda^2\omega^2(3-z) - v_2^2(1+z)}{(1+z)}\,; \tag{22}$$

$$C(v_2) = \frac{\lambda^2\omega^2 + v_2^2}{\lambda^2\omega^2 + v_2^2 - zB} \quad \text{et} \quad C_\varphi(v_2) = \frac{\lambda^2\omega^2 + v_2^2 - zB\cos^2\varphi}{\lambda^2\omega^2 + v_2^2 - zB}\,; \tag{23}$$

on a:

$$\alpha_2^{(1)} = -\frac{1}{AC}\frac{\partial f_2^{(0)}}{\partial v_2} \quad \text{et} \quad \frac{\Gamma_2^2}{6}\alpha_2^{(1)} C_\varphi = \frac{kT}{m_1\lambda}\frac{\partial f_2^{(0)}}{\partial v_2} + \frac{m_2 v_2}{m_1\lambda} f_2^{(0)}\,,$$

d'où

$$-\left(\frac{\Gamma_2^2 C_\varphi}{6AC} + \frac{kT}{m_1\lambda}\right)\frac{\partial f_2^{(0)}}{\partial v_2} = \frac{m_2 v_2}{m_1\lambda} f_2^{(0)}. \tag{24}$$

On en tire l'approximation d'ordre zéro $f_2^{(0)}$:

$$f_2^{(0)} = C' \exp\left[-\int_0^{v_2} \frac{6AC m_2 v_2\,\mathrm{d}v_2}{6ACkT + \Gamma_2^2 C_\varphi m_1\lambda}\right], \tag{25}$$

où C' doit être telle que:

$$\int_0^\infty f_2^{(0)}\,\mathrm{d}\boldsymbol{v}_2 = n_2\,.$$

3. – Discussion de l'équation (25).

1) Il est facile de montrer [4] qu'elle comporte, comme cas particulier, les résultats obtenus antérieurement par CHAPMAN et COWLING [3], MARGENAU [5], DRUYVESTEYN [6].

a) *Si* $\boldsymbol{H}_0$ *est perpendiculaire à* $\boldsymbol{E}$, on a d'après (23):

$$C_\varphi(v_2) = C(v_2)\,, \tag{26}$$

de sorte que $f_2^{(0)}$ devient:

$$f_2^{(0)} = C' \exp\left[-\int_0^{v_2} \frac{6m_2 v_2 A\,\mathrm{d}v_2}{6kTA + \Gamma_2^2 m_1 \lambda}\right], \tag{27}$$

expression que nous avons utilisé dans nos travaux antérieurs [4].

Si maintenant on suppose $\omega = 0$ (cas d'un champ électrique constant), les relations (20) et (21) s'écrivent

$$z(v_2) = \frac{e_2^2 H_0^2 \lambda^2}{m_2^2 v_2^2} \qquad \text{et} \qquad A(v_2) = \frac{v_2^2(1+z)}{\lambda}\,,$$

et l'équation (27) s'écrit alors

$$f_2^{(0)} = C' \exp\left[-\int_0^{v_2} \frac{6m_2 v_2^3(1+z)\mathrm{d}v_2}{6kTv_2^2(1+z) + \Gamma_2^2 m_1 \lambda^2}\right], \tag{27a}$$

on voit aisément qu'elle correspond à la formule donnée par CHAPMAN et COWLING, dans le cas où le terme en Γ_2^2 est prépondérant par rapport a celui en kT.

b) *Si* $H_0 = 0$, on voit de même que $z = 0$ et $A(v_2) = (\lambda^2\omega^2 + v_2^2)/\lambda$ d'où pour l'équation (27):

$$f_2^{(0)} = C' \exp\left[-\int_0^{v_2} \frac{m_2 v_2\,\mathrm{d}v_2}{kT + \dfrac{\Gamma_2^2 m_1 \lambda^2}{6(v_2^2 + \omega^2\lambda^2)}}\right]. \tag{27b}$$

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(4) R. JANCEL et T. KAHAN: *Compt. Rend. Acad. Sc.*, **236**, 788, 1478 et 2045 (1953); **237**, 1657 (1953); *Journ. Phys. et le Rad.*, **14**, 533 (1953); **15**, 26 (1954).

(5) H. MARGENAU: *Phys. Rev.*, **69**, 508 (1946).

(6) M. DRUYVESTEYN: *Physica*, **10**, 61 (1930); **1**, 1003 (1934).

On voit que (27*b*) est identique au résultat trouvé par MARGENAU [5] dans le cas d'un gas ionisé soumis seulement à l'action d'un champ électrique oscillant.

c) Si, dans (27*b*), on fait $\omega = 0$ et si l'on suppose de plus que kT est négligeable devant le terme en Γ_2^2, (27*b*) devient

$$f_2^{(0)} = C' \exp\left[-\int_0^{v_2} \frac{6m^2 v_2^3 \, \mathrm{d}v_2}{\Gamma_2^2 m_1 \lambda^2}\right]. \tag{27c}$$

Dans le cas où λ est indépendant de v_2, on voit que (27*c*) donne une répartition en $e^{-v_2^4}$ qui correspond aux notations près, à la formule donnée par DRUYVESTEYN [6] pour les gaz ionisés soumis à des champs électrostatiques intenses.

d) Si l'on suppose au contraire que, dans (25), le terme en kT est prépondérant devant le terme en Γ_2^2 (ce qui correspond soit au cas d'une intensité électrique faible, soit à des fréquences très élevées), on voit alors que $f_2^{(0)}$ est maxwellien et que la « température électronique » est égale à la température absolue du gaz T.

2) Il est intéressant de remarquer que l'expression de $f_2^{(0)}$ dépend essentiellement du libre parcours $\lambda(v_2)$. Or $\lambda(v_2)$ dépend de la loi d'intéraction entre électrons et molécules; en effet, on le calcule à partir des paramètres définissant une collision par les relations suivantes:

$$\lambda(v_2) = \frac{v_2}{2\pi n_1 \varphi_{12}(v_2)} \qquad \text{avec} \qquad \varphi_{12}(v_2) = \int_0^{\infty} (1 - \cos\chi) g b \, \mathrm{d}b \,,$$

où χ représente la variation angulaire de la vitesse relative durant une collision; généralement l'angle χ sera une fonction de la vitesse relative g et du paramètre d'impact b, fonction dont l'expression est déterminée par la loi d'intéraction.

Par conséquent, la fonction de répartition exacte des vitesses des électrons libres, à l'approximation d'ordre zero, dépend de la loi d'interaction entre les électrons et les molécules. Il est donc utile de calculer $\lambda(v_2)$ pour les gaz réels, soit théoriquement, soit à partir des résultats expérimentaux.

Signalons deux cas particuliers importants:

a) Cas des molécules sphériques et rigides, avec chocs élastiques; on a:

$$\varphi_{12}(v_2) = \tfrac{1}{2} \, r_2 \sigma_{12}^2 \qquad \text{avec} \qquad \sigma_{12} = \tfrac{1}{2}(\sigma_1 + \sigma_2) \,,$$

(où σ_1 et σ_2 sont les diamètres respectifs des deux sortes des particules). On obtient alors:

$$\lambda(v_2) = \frac{1}{\pi n_1 \sigma_{12}^2}, \tag{28a}$$

soit λ indépendant de v_2: Il est aisé, dans ce cas, de calculer (25) car on est ramené à l'intégration de fractions rationelles. Le résultat complet a été donné dans un travail antérieur (4) pour un champ $\boldsymbol{H}_0$ perpendiculaire a $\boldsymbol{E}$.

b) Cas où la force entre molécules et électrons varie comme $1/r^5$; le calcul montre que $\lambda(v_2)$ est proportionnel à v_2, de sorte que la fréquence de collision $v_2/\lambda = \nu$ est indépendente de v_2.

On montre alors facilement que $f_2^{(0)}$ est maxwellien, mais avec une «température électronique» $T' > T$. En effet si l'on pose:

$$\frac{v_2}{\lambda(v_2)} = \nu \qquad \text{(fréquence de collision)} \tag{29}$$

et

$$\frac{e_2 H_0}{m_2} = \omega_H \qquad \text{(gyrofréquence des électrons)} \tag{30}$$

on a

$$z = \frac{\omega_H^2}{\omega^2 + \nu^2}, \qquad AC = \lambda(\omega_2 + \nu^2), \tag{31}$$

et

$$C_\varphi = \frac{\nu^2(\omega^2+\nu^2+\omega_H^2)(\omega^2+\nu^2+\omega_H^2\cos^2\varphi) + \omega^2[(\omega^2+\nu^2)(\omega^2+\nu^2+\omega_H^2)}{\nu^2(\omega^2+\nu^2+\omega_H^2)^2 + \omega^2(\omega^2+\nu^2-\omega_H^2)^2} - \frac{\omega_H^2[3(\omega^2+\nu^2)-\omega_H^2]\cos^2\varphi}{\nu^2(\omega^2+\nu^2+\omega_H^2)^2 + \omega^2(\omega^2+\nu^2-\omega_H^2)^2} = h(\nu), \tag{32}$$

et $f_2^{(0)}$ s'écrit alors

$$f_2^{(0)} = n_2\left(\frac{\varrho}{\pi}\right)^{\frac{3}{2}} \exp\left[-\varrho v_2^2\right], \tag{33}$$

avec

$$\varrho = \frac{3(\omega^2+\nu^2)m_2}{\sigma(\omega^2+\nu^2)kT + \Gamma_2^2 h(\nu)m_1}.$$

On voit donc que $f_2^{(0)}$ est maxwellien avec une « température électronique » donnée par

$$T' = T\left(1 + \frac{m_1 \Gamma_2^2 h(\nu)}{6kT(\omega^2 + \nu^2)}\right). \tag{34}$$

La grandeur $h(\nu)/(\omega^2+\nu^2)$ tend vers zéro avec $1/\omega^2$, de sorte que le terme correctif défini par (34) devient négligeable pour les fréquences élevées.

4. – Première approximation et expression générale du courant.

1) Le courant se calcule à partir de la vitesse de diffusion $\boldsymbol{w}_2$ des électrons, d'après la relation:

$$I = n_2 e_2 \boldsymbol{w}_2 , \tag{35}$$

$\boldsymbol{w}_2$ s'exprimant en fonction de l'approximation d'ordre 1 de f_2, nous devons donc calculer les fonctions $\alpha_2^{(1)}$, $\beta_2^{(1)}$, $\xi_2^{(1)}$, $\eta_2^{(1)}$, $\gamma_2^{(1)}$ et $\delta_2^{(1)}$; celles ci s'obtiennent aisément à partir des équations (16′), (16″), (17′), (17″) (18′) et (18″), lorsque $f_2^{(0)}$ est connue. On a:

$$\alpha_2^{(1)} = \frac{6m_2 v_2 f_2^{(0)}}{U}, \qquad \beta_2^{(1)} = \frac{6m_2 C\lambda\omega[(\lambda^2\omega^2 + v_2^2)(1-z) - zB']f_2^{(0)}}{(1+z)(\lambda^2\omega^2 + v_2^2)U}; \tag{36}$$

$$\xi_2^{(1)} = \frac{6e_2\lambda C f_2^{(0)}}{(\lambda^2\omega^2 + v_2^2)U}\left[\lambda^2\omega^2\left(\frac{1-z}{1+z}\right) - v_2^2\right], \quad \eta_2^{(1)} = -\frac{12e_2\lambda^2\omega C v_2 f_2^{(0)}}{(\lambda^2\omega^2 + v_2^2)(1+z)U}. \tag{37}$$

$$\gamma_2^{(1)} = -\frac{6m_2 v_2 z B f_2^{(0)}}{H_0^2(\lambda^2\omega^2 + v_2^2 - zB)U}, \quad \delta_2^{(1)} = -\frac{6m_2\lambda\omega z B' C f_2^{(0)}}{H_0^2(1+z)(\lambda^2\omega^2 + v_2^2)U}, \tag{38}$$

avec

$$B' = \lambda^2\omega^2(1-z) - v_2^2(3+z) \quad \text{et} \quad U = 6ACkT + \Gamma_2^2 C\varphi m_1\lambda . \tag{39}$$

Ces équations permettent de calculer la 1ère approximation de la fonction de distribution telle que nous l'avons définie dans (6); nous pouvons d'ailleurs l'écrire sous la forme:

$$f_2 = f_2^{(0)} + f_2^{(1)} . \tag{6′}$$

Nous avons déja vu, au paragraphe précedent, que $f_2^{(0)}$ est maxwellien si le terme en Γ_2^2 est négligeable devant celui en kT. Nous allons montrer que, dans ce cas, $f_2^{(1)}$ tend aussi vers la 1ère approximation du cas maxwellien; en

effet, si nous supposons que

$$\frac{\Gamma_2^2 C\varphi m_1\lambda}{6HCkT} \ll 1\,, \tag{40}$$

on a alors $U \cong 6ACkT$ et l'application des formules (6), (36), (37), et (38) donne:

$$\begin{aligned} f_2^{(1)} = \frac{f_2^{(0)}}{AkT}\Bigg\{ & \left[\frac{m_2 v_2}{C}\cos\omega t - \frac{\omega m_2\lambda[\lambda^2\omega^2(1-z)^2+v_2^2(1+z)^2]}{(1+z)(\lambda^2\omega^2+v_2^2)}\sin\omega t\right](\boldsymbol{\Gamma}_2\cdot\boldsymbol{v}_2) - \\ & + \left[\frac{e_2\lambda[\lambda^2\omega^2(1-z)-v_2^2(1+z)}{\lambda^2\omega^2-v_2^2)(1-z)}\cos\omega t - \frac{2\omega e_2\lambda^2 v_2}{(\lambda^2\omega^2-v_2^2)(1+z)}\sin\omega t\right](\boldsymbol{H}_0\wedge\boldsymbol{\Gamma}_2)\cdot\boldsymbol{v}_2 - \\ & - \left[\frac{m_2 e_2 z[\lambda^2\omega^2(3-z)-v_2^2(1+z)]}{CH_0^2[\lambda^2\omega^2(1-z)^2+v_2^2(1-z^2)]}\cos\omega t - \frac{\omega m_2 z\lambda[\lambda^2\omega^2(1-z)-v_2^2(3+z)]}{H_0^2(1+z)(\lambda^2\omega^2+v_2^2)}\sin\omega\right| \\ & \cdot[\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\boldsymbol{\Gamma}_2)]\cdot\boldsymbol{v}_2\Bigg\} = [f_2^{(1)}]_{\text{Maxwell}}\,, \end{aligned} \tag{41}$$

et nous retrouvons, aux notations près, certains résultats de CHAPMAN et COWLING [3] dans la théorie maxwellienne [7].

2) D'après la definition de $\boldsymbol{w}_2$, le courant $\boldsymbol{I}$ s'écrit:

$$\begin{aligned} \boldsymbol{I} = n_2 e_2 \boldsymbol{w}_2 = \frac{4\pi n_2 e_2}{3}\Bigg\{ & \left[\frac{1}{n_2}\int_0^\infty(\alpha_2^{(1)}\cos\omega t+\beta_2^{(1)}\sin\omega t)v_2^4\,\mathrm{d}v_2\right]\boldsymbol{\Gamma}_2 + \\ & + \left[\frac{1}{n_2}\int_0^\infty(\xi_2^{(1)}\cos\omega t+\eta_2^{(1)}\sin\omega t)v_2^4\,\mathrm{d}v_2\right](\boldsymbol{H}_0\wedge\boldsymbol{\Gamma}_2) + \\ & + \left[\frac{1}{n_2}\int_0^\infty(\gamma_2^{(1)}\cos\omega t+\delta_2^{(1)}\sin\omega t)v_2^4\,\mathrm{d}v_2\right]\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\boldsymbol{\Gamma}_2)\Bigg\}\,. \end{aligned} \tag{42}$$

Nous écrirons cette formule:

$$\boldsymbol{I} = \frac{4\pi n_2 e_2}{3m_2}[J_1\boldsymbol{E} + J_2(\boldsymbol{H}_0\wedge\boldsymbol{E}) + J_3\boldsymbol{H}_0\wedge(\boldsymbol{H}_0\wedge\boldsymbol{E})]\,, \tag{42'}$$

avec les définitions suivantes:

$$J_1 = \frac{1}{n_2}\int_0^\infty(\alpha_2^{(1)}\cos\omega t+\beta_2^{(1)}\sin\omega t)v_2^4\,\mathrm{d}v_2 = 6m_2(G_1\cos\omega t+\omega G_1'\sin\omega t)\,; \tag{43}$$

$$J_2 = \frac{1}{n_2}\int_0^\infty(\xi_2^{(1)}\cos\omega t-\eta_2^{(1)}\sin\omega t)v_2^4\,\mathrm{d}v_2 = 6e_2(G_2\cos\omega t-2\omega G_2'\sin\omega t)\,; \tag{44}$$

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[7] R. JANCEL et T. KAHAN: *Compt. Rend. Acad. Sc.*, **238**, 995 (1954); *Journ. Phys. et le Rad.*, lettre à l'éditeur, sous presse.

$$(45) \qquad J_3 = \frac{1}{n_2}\int_0^\infty (\gamma_2' \cos\omega t + \delta_2^{(1)} \sin\omega t) r_2^4 \, dv_2 = -\frac{6m_2}{H_0^2}(G_3 \cos\omega t + \omega G_3' \sin\omega t) .$$

L'expression des intégrales G_1, G_1', G_2, G_2', G_3 et G_3' s'obtient immédiatement à partir des formules (36), (37) et (38).

Les propriétés électriques du gaz ionisé sont donc décrites par $f_2^{(0)}$ et les 6 intégrales précédentes, et l'étude numérique, au moins approximative, de ces expressions est nécessaire à la connaissance des grandeurs (conductivité par exemple) qui déterminent la propagation des ondes électromagnetiques dans un tel milieu.

Nous allons écrire $\boldsymbol{I}$ dans le cas où $\boldsymbol{H}_0$ est perpendiculaire à $\boldsymbol{E}$, cette expressions devant nous être utile dans la suite. On trouve facilement:

$$(46) \qquad \boldsymbol{I} = \frac{4\pi n_2 e_2^2}{3m_2}[(J_1 - H_0^2 J_3)\boldsymbol{E} + J_2(\boldsymbol{H}_0 \wedge \boldsymbol{E})] .$$

Prenons des axes tels que Ox soit parallèle à $\boldsymbol{E}$, Oy à $(\boldsymbol{H}_0 \wedge \boldsymbol{E})$, Oz complétant le trièdre; les composantes de $\boldsymbol{E}$ et $\boldsymbol{H}_0$ sont alors respectivement $(E, 0, 0)$ et $(0, 0, H_0)$. Les composantes de $\boldsymbol{I}$ sont alors données par:

$$I_x = \mu[(G_1 + G_3)\cos\omega t + \omega(G_1' + G_3')\sin\omega t]\boldsymbol{E}$$
$$I_y = \mu(G_2 \cos\omega t - 2\omega G_2' \sin\omega t)\omega_H E$$
$$I_z = 0$$

où l'on a posé $\mu = 8\pi n_2 e_2^2$.

Avant d'appliquer ces résultats, nous allons calculer la valeur de $\boldsymbol{I}$ dans deux cas particuliers.

5. – Valeur de $\boldsymbol{I}$ dans des cas particuliers.

a) La frequence de collision $\nu = v^2/\lambda$ est indépendente de v_2. – Nous avons vu que dans ce cas $f_2^{(0)}$ est maxwellien et s'exprime par (34): les 6 intégrales G se calculent alors aisément en fonction de ν. Nous écrivons le résultat pour simplifier dans le cas où $\boldsymbol{H}_0$ est perpendiculaire à $\boldsymbol{E}$; nous avons alors en posant

$$(48) \qquad g(\nu) = \frac{\omega^2(\omega^2 + \nu^2 - \omega_H^2)^2 + \nu^2(\omega^2 + \nu^2 + \omega_H^2)^2}{(\omega^2 + \nu^2)(\omega^2 + \nu^2 + \omega_H^2)} ,$$

et en utilisant (46):

$$(49) \qquad \boldsymbol{I} = \frac{n_2 e_2^2}{m_2 g(\nu)}\left\{\left(\nu\cos\omega t + \omega\frac{\omega^2 + \nu^2 - \omega_H^2}{\omega^2 + \nu^2 + \omega_H^2}\sin\omega t\right)\boldsymbol{E} + \right.$$
$$\left. + \frac{e_2}{m_2}\left[\frac{1}{\omega^2 + \nu^2}\left(\omega^2\frac{\omega^2 + \nu^2 - \omega_H^2}{\omega^2 + \nu^2 + \omega_H^2} - \nu^2\right)\cos\omega t - \frac{2\omega\nu}{\omega^2 + \nu^2 + \omega_H^2}\sin\omega t\right](\boldsymbol{H}_0 \wedge \boldsymbol{E})\right\} .$$

b) Cas où $H_0 = 0$, ω et E étant petits. – Plus précisément, nous supposons ici que ω^2 et Γ_2^2 sont négligeables. La condition $H_0 = 0$ donne $z = 0$; d'autre part:

$$AC = \frac{\lambda^2\omega^2 + v_2^2}{\lambda},$$

et pour ω suffisamment petit, on peut poser:

$$AC \cong \frac{v_2^2}{\lambda}.$$

De plus Γ_2^2 étant négligeable, on voit immédiatement d'après (25) que $f_2^{(0)}$ est maxwellien, soit:

$$f_2^{(0)} = n_2 \left(\frac{m_2}{2\pi kT}\right)^{\frac{3}{2}} \exp\left[-\frac{m_2 v_2^2}{2kT}\right].$$

Avec ces hypothèses, on peut alors écrire

$$G_1 \cong \frac{\lambda}{6kTn_2}\int_0^\infty v_2^3 f_2^{(0)}\,\mathrm{d}v_2 = \frac{\lambda}{6\sqrt{2m_2kT}}\left(\frac{1}{\pi}\right)^{\frac{3}{2}},$$

et

$$G_2 \cong \frac{\lambda^2}{6kTn_2}\int v_2^2 f_2^{(0)}\,\mathrm{d}v^2 = \frac{\lambda^2}{24\pi kT}.$$

Le courant s'écrit donc:

$$I = \frac{4n_2e_2^2\lambda E}{3\sqrt{2m_2kT}}\cos\omega t + \frac{\omega n_2 e_2^2\lambda^2 E}{3kT}\sin\omega t. \tag{50}$$

Le premier membre de cette formule représente la formule de mobilité des ions de Langevin [8]. Celle-ci est donc un cas particulier de *la formule* (47) (1ère *équation*) *que l'on peut considérer comme une forme généralisée de la formule de Langevin, dans le cas où l'on a un champ magnétique* $\boldsymbol{H}_0$ *et où* ω *et* $\boldsymbol{E}$ *sont quelconques.*

[8] P. Langevin: *Ann. Chim. Phys.*, **27**, 317 (1903).

6. – Applications diverses.

Nous allons maintenant utiliser les résultats précédents au calcul du tenseur de conductivité électrique, du tenseur diélectrique du gaz et du coefficient de Hall.

a) Tenseur de conductivité. – Utilisons la formule (42′) en prenant des axes tels que $\boldsymbol{E}$ et $\boldsymbol{H}_0$ aient respectivement les composantes (E_x, E_y, E_z) et $(0, -H_0 \sin\varphi, H_0 \cos\varphi)$; le champ $\boldsymbol{H}_0$ fait un angle φ avec l'axe Oz et se trouve dans le plan (yOz). Nous avons alors:

$$(51)\quad \begin{cases} I_x = \dfrac{4\pi n_2 e_2^2}{3m_2}[(J_1 - H_0^2 J_3)E_x - J_2 H_0 E_y \cos\varphi - J_2 H_0 E_z \sin\varphi]\,; \\ I_y = \dfrac{4\pi n_2 e_2^2}{3m_2}[J_2 H_0 E_x \cos\varphi + (J_1 - H_0^2 J_3 \cos^2\varphi)E_y - J_3 H_0^2 E_z \sin\varphi \cos\varphi]\,; \\ I_z = \dfrac{4\pi n_2 e_2^2}{3m_2}[J_2 H_0 E_x \sin\varphi - J_3 H_0^2 E_y \sin\varphi \cos\varphi + (J_1 - H_0^2 J_3 \sin^2\varphi)E_z]\,. \end{cases}$$

Le tenseur de conductivité étant défini par la relation

$$I_\mu = \sum_\nu \sigma_{\mu\nu} E_\nu$$

ses composantes ont pour expression:

$$(52)\quad \begin{cases} \sigma_{xx} = \mu[(G_1 + G_3)\cos\omega t + \omega(G_1' + G_3')\sin\omega t]; \\ \sigma_{xy} = -\sigma_{yx} = -\mu\omega_H \cos\varphi(G_2 \cos\omega t - 2\omega G_2' \sin\omega t)\,; \\ \sigma_{xz} = -\sigma_{zx} = -\mu\omega_H \sin\varphi(G_2 \cos\omega t - 2\omega G_2' \sin\omega t)\,; \\ \sigma_{yy} = \mu[(G_1 + G_3\cos^2\varphi)\cos\omega t + \omega(G_1' + G_3'\cos^2\varphi)\sin\omega t]\,; \\ \sigma_{yz} = \sigma_{zy} = \mu \sin\varphi \cos\varphi(G_3 \cos\omega t + \omega G_3' \sin\omega t)\,; \\ \sigma_{zz} = \mu[(G_1 + G_3 \sin^2\varphi)\cos\omega t + \omega(G_1' + G_3' \sin^2\varphi)\sin\omega t]\,. \end{cases}$$

b) Tenseur diélectrique. – Il est défini par la relation $\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}$ avec $\boldsymbol{I} = \mathrm{d}\boldsymbol{P}/\mathrm{d}t$ et $D_\mu = \sum_\nu \varepsilon_{\mu\nu} E_\nu$.

Les composantes $\varepsilon_{\mu\nu}$ se calculent facilement à l'aide de (52) et des définitions précédentes. On a:

$$
(53)\quad
\begin{cases}
\varepsilon_{xx} = 1 + 4\pi\mu\left[\dfrac{G_1 + G_3}{\omega}\sin\omega t - (G_1' + G_3')\cos\omega t\right]; \\
\varepsilon_{xy} = -\varepsilon_{yx} = -\dfrac{4\pi\mu\omega_H\cos\varphi}{\omega}(G_2\sin\omega t + 2\omega G_2'\cos\omega t); \\
\varepsilon_{xz} = -\varepsilon_{zx} = -\dfrac{4\pi\mu\omega_H\sin\varphi}{\omega}(G_2\sin\omega t + 2\omega G_2'\cos\omega t); \\
\varepsilon_{yy} = 1 + 4\pi\mu\left[\dfrac{G_1 + G_3\cos^2\varphi}{\omega}\sin\omega t - (G_1' + G_3'\cos^2\varphi)\cos\omega t\right]; \\
\varepsilon_{yz} = \varepsilon_{zy} = 4\pi\mu\sin\varphi\cos\varphi\left(\dfrac{G_3}{\omega}\sin\omega t - G_3'\cos\omega t\right); \\
\varepsilon_{zz} = 1 + 4\pi\mu\left[\dfrac{G_1 + G_3\sin^2\varphi}{\omega}\sin\omega t - (G_1' + G_3'\sin^2\varphi)\cos\omega t\right].
\end{cases}
$$

Remarque: Ces expressions se réduisent, pour le cas où $\boldsymbol{H}_0$ est faible (ou est perpendiculaire à $\boldsymbol{E}$) à des formules que nous avons établies et utilisées dans des travaux antérieurs ([4]). Les tenseurs (52) et (53) sont donc la généralisation naturelle de ces formules pour des champs $\boldsymbol{H}_0$ d'intensité et d'orientation quelconques.

c) Coefficient de Hall. – Pour l'obtenir nous nous plaçons dans le cas où $\boldsymbol{E}$ et $\boldsymbol{H}_0$ sont perpendiculaires; nous prenons à cet effet $E_z = 0$ et $\varphi = 0$. On voit alors d'après (51) que $I_z = 0$; faisons alors tourner les axes de telle sorte que $I_y = 0$, E_y et E_x sont alors liés par la relation

$$E_y = -\frac{J_2}{(J_1 - H_0^2 J_3)} H_0 E_x,$$

d'où, d'après le 1ère équation de (51)

$$I_x = I = -\frac{4\pi n_2 e_2^2}{3m_2}\left[\frac{(J_1 - H_0^2 J_3)^2 + H_0^2 J_2^2}{H_0 J_2}\right],$$

et le coefficient de Hall s'écrit:

$$(54)\qquad R = \frac{E_y}{I} = -\frac{3m_2 H_0 J_2}{4\pi n_2 e_2^2}\left[\frac{1}{(J_1 - H_0^2 J_3)^2 + H_0^2 J_2^2}\right].$$

Dans le cas où $\omega = 0$, on a $J_1 = 6m_2G_1$, $J_2 = 6e_2G_2$ et $J_3 = -(6m_2/H_0^2)G_3$ et

$$R = -\frac{\omega_H G_2}{\mu}\left[\frac{1}{(G_1+G_3)^2+\omega_H^2 G_2^2}\right].$$

Nous appliquerons ce résultat au cas particulier où $f_2^{(0)}$ est maxwellien et en supposant que

$$\frac{\nu}{\omega_H} \gg 1\,, \quad \left(\text{avec } \nu = \frac{\bar{v}_2}{\lambda}\right).$$

Il vient, en négligeant les termes en Γ_2^2

$$G_1 = \frac{1}{n_2}\int_0^\infty \frac{v_2^5 f_2^{(0)}\,\mathrm{d}v_2}{6\left(\frac{\lambda^2\omega^2+v_2^2}{\lambda}\right)kT} = \frac{\lambda}{6kTn_2}\int_0^\infty v_2^3 f_2^{(0)}\,\mathrm{d}v_2 = \frac{\lambda\bar{v}_2}{24\pi kT}\,;$$

$$G_3 = -\frac{z}{(1+z)n_2}\int_0^\infty \frac{v_2^5 f_2^{(0)}\,\mathrm{d}v_2}{6ACkT} = -\frac{z\lambda\bar{v}_2}{24\pi kT(1+z)}\,,$$

d'où

$$G_1+G_3 = \frac{\lambda\bar{v}_2}{24\pi kT}\,\frac{1}{1+z}\,.$$

Or, d'après l'hypothèse (55), z est négligeable; en effet

$$z = \frac{\omega_H^2\lambda^2}{v_2^2} \cong \frac{\omega_H^2}{\nu^2} \ll 1\,.$$

On a donc

$$G_1+G_3 = \frac{\lambda\bar{v}_2}{24\pi kT}\,.$$

On a de même

$$G_2 = -\frac{\lambda^2}{6kTn_2}\int_0^\infty v_2^2 f_2^{(0)}\,\mathrm{d}v_2 = -\frac{\lambda^2}{24\pi kT}\,.$$

Il vient alors:

$$(G_1+G_3)^2+\omega_H^2G_2^2 = \frac{\lambda^2(\bar{v}_2)^2}{(24\pi kT)^2}\left[1+\frac{\omega_H^2\lambda^2}{(\bar{v}_2)^2}\right] \cong \frac{\lambda^2(\bar{v}_2)^2}{(24\pi kT)^2}\,,$$

d'où:

$$R = \frac{\omega_H}{n_2 e_2^2} \frac{3kT}{(\bar{r}_2)^2},$$

et, en remarquant que

$$(\bar{r}_2)^2 = \frac{8kT}{\pi m_2},$$

on obtient finalement

$$R = \frac{3\pi}{8} H_0 \frac{1}{n_2 e_2}. \tag{55}$$

d) *Déviation d'un faisceau électronique.* – Nous-nous plaçons toujours dans le cas où $\boldsymbol{E}$ et $\boldsymbol{H}_0$ sont perpendiculaires; nous appliquons donc les formules (47) qui donnent pour la déviation instantanée

$$d = \frac{I_y}{I_x} = \omega_H \frac{G_2 \cos \omega t - 2\omega G_2' \sin \omega t}{[(G_1 + G_3) \cos \omega t + \omega(G_1' + G_3') \sin \omega t]},$$

et l'on peut calculer facilement, à partir de cette formule, la déviation moyenne.

7. – Discussion et comparaison avec la méthode du libre parcours moyen.

Cette méthode développée en particulier par Townsend, L. G. H. Huxley (9) et M. Bayet (10) consiste à calculer le déplacement subi par l'électron sous l'influence du champ, durant l'intervalle compris entre deux chocs successifs. Le déplacement moyen d'un ensemble d'électrons par unité de temps fournit la vitesse de diffusion des électrons à travers le gaz. D'une manière générale, on suppose toujours que l'intensité du champ est faible; on effectue le calcul en admettant d'abord que les électrons ont tous la même vitesse, puis on prend la valeur moyenne du résultat avec une fonction de répartition des vitesses électroniques.

(9) L. G. H. Huxley: *Phil. Mag.*, **23**, 210 (1937a); **23**, 442 (1937b); **29**, 313 (1940); *Proc. Phys. Soc.*, **68**, 884 (1951b).

(10) M. Bayet: *Journ. Phys. et le Rad.*, **13**, 579 (1952).

En raison de sa nature même, cette méthode présente l'inconvenient de ne s'appliquer qu'à des champs faibles, sans qu'il soit possible de préciser d'une manière rigoureuse la validité et l'ordre de grandeur des approximations mises en jeu en vertu de cette hypothèse. De plus, le fait de conduire le calcul en supposant d'abord que tous les électrons ont la même vitesse après un choc entraine des divergences importantes avec la théorie des distributions comme l'a souligné COWLING [11].

On peut remarquer aussi que les expressions moyennes obtenues par ces auteurs (en particulier HUXLEY) n'ont de sens que si l'on connait la fréquence des chocs ν en fonction de la vitesse v_2, ainsi que la fonction de distribution des vitesses; sans cette précision il est impossible d'évaluer les grandeurs moyennes qui déterminent la vitesse de diffusion .

HUXLEY a montré que cette méthode conduit à la formule

$$w_2 = \frac{2}{3}\frac{e_2 E}{m_2}\frac{\lambda}{v_2},$$

pour la vitesse de diffusion en présence d'un champ électrique constant (avec $H = 0$). Cet auteur semble attacher grande importance au facteur $2/3$, bien qu'il ne précise pas la nature de la fonction de distribution qui détermine v_2. Nous allons voir que nous sommes conduits à remplacer ce facteur par $8/3\pi$, dans le cas d'une distribution maxwellienne.

Afin de faciliter la comparaison de ces formules basées sur la notion du libre parcours moyen avec celles obtenues à partir de l'équation de Boltzmann, nous traiterons deux cas particuliers:

a) *Cas où* $H_0 = 0$, $\omega = 0$. – Nous supposerons de plus que Γ_2^2 est négligeable ce qui revient à supposer que l'intensité du champ $\boldsymbol{E}$ est suffisamment faible. Dans ce cas, nous avons déjà vu que $f_2^{(0)}$ est maxwellien et que, si l'on suppose λ indépendant de v_2,

$$G_1 = \frac{\lambda}{6kTn_2}\int_0^\infty v_2^3 f_2^{(0)}\,\mathrm{d}v_2 = \frac{\lambda\bar{v}_2}{24\pi kT} = \frac{\lambda\bar{v}_2}{8\pi m_2\overline{v_2^2}}.$$

On a donc

$$\boldsymbol{I} = \frac{n_2 e_2^2 \lambda \bar{v}_2}{m_2 \overline{v_2^2}}\boldsymbol{E},$$

et comme

$$\sqrt{\overline{v_2^2}} = \sqrt{\frac{3}{8}\pi}\,\bar{v}_2,$$

(11) T. COWLING: *Proc. Roy. Soc.*, **A 183**, 453 (1944).

dans le cas d'une répartition maxwellienne, il vient:

$$\boldsymbol{I} = n_2 e_2 \frac{8}{3\pi} \frac{e_2 \boldsymbol{E}}{m_2} \frac{\lambda}{\bar{v}_2}, \tag{56}$$

d'où la vitesse de diffusion

$$\boldsymbol{w}_2 = \frac{8}{3\pi} \frac{e_2 \boldsymbol{E}}{m_2} \frac{\lambda}{\bar{v}_2}. \tag{57}$$

On voit bien que, dans ce cas, le coefficient de 2/3 est remplacée par $8/3\pi$.

b) Cas où ν est indépendant de v_2. – Nous utiliserons la formule (49) (où $\boldsymbol{H}_0$ est perpendiculaire à $\boldsymbol{E}$) et nous l'appliquerons à deux cas:

1) $\boldsymbol{H}_0 =$ et $\omega = 0$; on a alors:

$$\omega_H = 0 \quad \text{et} \quad g(v) = v^2,$$

d'où

$$G_1 = \frac{1}{8\pi m_2 \nu}.$$

Il vient donc:

$$\boldsymbol{I} = n_2 e_2 \frac{e_2 \boldsymbol{E}}{m_2} \frac{1}{\nu}, \tag{58}$$

formule du même type que (56) avec un coefficient égal à 1. Il faut cependant remarquer que, dans ce cas, la « température électronique » peut être différente de T, car on ne fait alors aucune hypothèse sur l'intensité de $\boldsymbol{E}$.

2) $\omega = 0$ et $\omega_H \neq 0$. On a alors $g(v) = v^2 + \omega_H^2$, d'où, en appliquant simultanément (47) et (49), on obtient:

$$I_x = n_2 e_2 \frac{e_2 E}{m_2} \frac{\nu}{\nu^2 + \omega_H^2}, \tag{59}$$

$$I_y = - n_2 e_2 \frac{e_2 E}{m_2} \frac{\omega_H}{\nu^2 + \omega_H^2}. \tag{60}$$

On constate que ces deux résultats sont équivalents à ceux établis primitivement par TOWNSEND: cela s'explique aisément en remarquant que TOWNSEND supposait que la vitesse v_2 était la même pour tous les électrons et que λ était indépendant de v_2, de sorte que $\nu = v_2/\lambda$ ne dépendait pas de v_2. Mais il faut remarquer, comme précédemment, que ces formules peuvent être valables pour des « températures électroniques » différentes de T; de plus, elles ne peuvent s'appliquer que pour le cas particulier d'une loi d'interaction électron-composante lourde en $1/r^5$.

8. – Équations de propagation des ondes électromagnétiques planes dans un gaz ionisé.

Nous allons étudier la propagation des ondes électromagnétiques dans un gas ionisé dont la perméabilité magnétique est supposée égale à celle du vide et dont les propriétés diélectriques sont déterminées par un tenseur de la forme (53). Si l'on désigne par $\boldsymbol{I}$ la densité de courant, correspondant au déplacement des charges électriques du milieu sous l'action du champ électromagnétique, les équations de Maxwell s'écrivent:

$$\text{rot}\,\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}\,, \tag{61}$$

$$\text{rot}\,\boldsymbol{H} = \frac{1}{c}\left(\frac{\partial \boldsymbol{E}}{\partial t} + 4\pi \boldsymbol{I}\right), \tag{62}$$

avec $\text{div}\,\boldsymbol{E} = 4\pi\varrho$ et $\text{div}\,\boldsymbol{H} = 0$ (dans ces équations on emploie les unités électromagnétiques pour les champs magnétiques et les unités électrostatiques pour les champs, courants et charges électriques). Posons:

$$\boldsymbol{I} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} \qquad \text{et} \qquad \boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}\,, \tag{63}$$

(où $\boldsymbol{P}$ désigne la polarisation électrique du milieu)

L'équation (62) s'écrit alors:

$$\text{rot}\,\boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}\,, \tag{64}$$

et, en combinant convenablement les équations (61), (63) et (64), on obtient

les équations de propagation

$$\text{rot rot } \boldsymbol{E} = \text{grad div } \boldsymbol{E} - \Delta \boldsymbol{E} = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{D}}{\partial t^2}\,, \tag{65}$$

$$\Box \boldsymbol{H} = \frac{4\pi}{c}\frac{\partial}{\partial t}\text{rot } \boldsymbol{P}\,. \tag{66}$$

Pour étudier les propriétés des ondes planes monochromatiques, nous chercherons des solutions des équations (65) et (66) de la forme:

$$\boldsymbol{F} = \boldsymbol{F}_0 \exp\left[i\omega\left(t - \frac{\boldsymbol{N}\cdot\boldsymbol{r}}{u}\right)\right], \tag{67}$$

et nous supposerons que les vecteurs $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$ et $\boldsymbol{P}$ sont tous de ce type. Dans cette expression u est complexe, sa partie réelle representant la vitesse de phase de l'onde, sa partie imaginaire donnant un facteur d'atténuation des ondes, $\boldsymbol{N}$ est la normale au plan d'onde, $\boldsymbol{r}$ le rayon vecteur du point courant.

D'après (67) on calcule facilement les expressions de rot rot $\boldsymbol{F}$, $\partial^2\boldsymbol{F}/\partial t^2$, $\Box\boldsymbol{F}$ qui permettent d'écrire les équations de propagation sous la forme

$$\frac{c^2}{u^2}\boldsymbol{E} - \boldsymbol{D} - \frac{c^2}{u^2}(\boldsymbol{E}\cdot\boldsymbol{N})\boldsymbol{N} = 0\,, \tag{68}$$

$$\left(\frac{c^2}{u^2} - 1\right)\boldsymbol{H} = \frac{4\pi c}{u}(\boldsymbol{N}\wedge\boldsymbol{P})\,. \tag{69}$$

Ces équations déterminent complètement la propagation d'une onde plane, s l'on connaît la relation entre $\boldsymbol{D}$ et $\boldsymbol{E}$ qui caractérise le milieu considéré. Sans connaître le tenseur $\varepsilon_{\mu\nu}$, on peut déduire de (68) et (69) diverses propriétés (voir [4]):

1) Le vecteur $\boldsymbol{D}$ est dans le plan d'onde.

2) Le vecteur $\boldsymbol{H}$ est dans le plan d'onde et perpendiculaire au plan $(\boldsymbol{D}, \boldsymbol{E})$. Il s'ensuit que le plan d'onde est défini par les vecteurs orthogonaux $\boldsymbol{D}$ et $\boldsymbol{H}$ qui forment avec $\boldsymbol{N}$ un trièdre direct. Le champ électrique $\boldsymbol{E}$ est situé dans le plan $(\boldsymbol{D}, \boldsymbol{N})$ ainsi que le vecteur de Poynting: $\boldsymbol{S} = (c/4\pi)(\boldsymbol{E}\wedge\boldsymbol{H})$. Enfin l'angle $(\widehat{\boldsymbol{N}, \boldsymbol{S}})$ est égal à l'angle $(\widehat{\boldsymbol{D}, \boldsymbol{E}})$.

Pour étudier les autres propriétés de l'onde plane, il faut porter dans (68) l'expression du tenseur $\varepsilon_{\mu\nu}$. Nous supposerons que le milieu considéré peut être décrit par la théorie magnéto-ionique précédente et que le tenseur diélectrique est donné par (53). Nous utiliserons ces formules sous la forme

complexe correspondante, soit:

$$
(70)\quad
\left\{
\begin{aligned}
&\varepsilon'_{xx} = 1 - 4\pi\mu\left[(G'_1 + G'_3) + i\,\frac{G_1 + G_3}{\omega}\right] = \varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}}\,,\\
&\varepsilon'_{xy} = -\,\varepsilon'_{yx} = -\,4\pi\mu\omega_H\left(2G'_2 - i\,\frac{G_2}{\omega}\right)\cos\varphi = \varepsilon_{\mathrm{II}}\cos\varphi\,,\\
&\varepsilon'_{xz} = -\,\varepsilon'_{zx} = -\,4\pi\mu\omega_H\left(2G'_2 - i\,\frac{G_2}{\omega}\right)\sin\varphi = \varepsilon_{\mathrm{II}}\sin\varphi\,,\\
&\varepsilon'_{yy} = 1 - 4\pi\mu\left[(G'_1 + G'_3\cos^2\varphi) + i\,\frac{G_1 + G_3\cos^2\varphi}{\omega}\right] = \varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}}\cos^2\varphi\,,\\
&\varepsilon'_{yz} = \varepsilon'_{zy} = -\,4\pi\mu\left(G'_3 + i\,\frac{G_3}{\omega}\right)\sin\varphi\,\cos\varphi = \varepsilon_{\mathrm{III}}\sin\varphi\,\cos\varphi\,,\\
&\varepsilon'_{zz} = 1 - 4\pi\mu\left[(G'_1 + G'_3\sin^2\varphi) + i\,\frac{(G_1 + G_3\sin^2\varphi)}{\omega}\right] = \varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}}\sin^2\varphi\,,
\end{aligned}
\right.
$$

où l'on a posé

$$
(71)\quad
\left\{
\begin{aligned}
&\varepsilon_{\mathrm{I}} = 1 - 4\pi\mu\left(G'_1 + i\,\frac{G_1}{\omega}\right);\\
&\varepsilon_{\mathrm{II}} = -\,4\pi\mu\omega_H\left(2G'_2 - i\,\frac{G_2}{\omega}\right);\\
&\varepsilon_{\mathrm{III}} = -\,4\pi\mu\left(G'_3 + i\,\frac{G_3}{\omega}\right).
\end{aligned}
\right.
$$

(Nous avons ici, du fait du choix des unités, $\omega_H = e_2 H_0/m_2 c$.

Par définition, (70) établit la relation entre $\boldsymbol{D}$ et $\boldsymbol{E}$, et en portant celle-ci dans l'équation (68), on peut calculer les indices de réfraction correspondant aux différents modes de propagation des ondes planes et établir ainsi la biréfringence du plasma considéré.

9. – Indices de réfraction, atténuation, vitesses de phase et de groupe.

Nous supposerons que le champ $\boldsymbol{H}_0$ fait un angle φ avec la normale à l'onde $\boldsymbol{N}$, et nous prendrons celle-ci comme axe Oz; on peut admettre sans diminuer la généralité que $\boldsymbol{H}_0$ est dans le plan yOz et qu'il a pour composantes $(0, -H_0\sin\varphi, H_0\cos\varphi)$. Avec ces conventions, l'équation (68) s'écrit, en uti-

lisant les proprietés de symétrie de (70):

$$(72)\qquad \begin{cases} \left(\dfrac{c^2}{u^2} - \varepsilon'_{xx}\right) E_x - \varepsilon'_{xy} E_y - \varepsilon'_{xz} E_z = 0\,, \\ \varepsilon'_{xy} E_x + \left(\dfrac{c^2}{u^2} - \varepsilon'_{yy}\right) E_y - \varepsilon'_{yz} E_z = 0\,, \\ \varepsilon'_{xz} E_x - \varepsilon'_{yz} E_y - \varepsilon'_{zz} E_z = 0\,. \end{cases}$$

Nous avons un système homogène et linéaire en E_x, E_y, E_z qui n'a des solutions que si son déterminant est nul, soit:

$$(73)\qquad D = \begin{vmatrix} \dfrac{c^2}{u^2} - \varepsilon'_{xx} & -\varepsilon'_{xy} & -\varepsilon'_{xz} \\ \varepsilon'_{xy} & \left(\dfrac{c^2}{u^2} - \varepsilon'_{yy}\right) & -\varepsilon'_{yz} \\ \varepsilon'_{xz} & -\varepsilon'_{yz} & -\varepsilon'_{zz} \end{vmatrix} = 0\,.$$

On obtient en développant ce déterminant, une équation bicarrée en c/u qui s'écrit, en utilisant (71):

$$(74)\qquad (\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}} \sin^2\varphi)\frac{c^4}{u^4} - [(\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}})(2\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}} \sin^2\varphi) + \varepsilon_{\mathrm{II}}^2 \sin^2\varphi]\frac{c^2}{u^2} + \\ + \varepsilon_{\mathrm{I}}[(\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}})^2 + \varepsilon_{\mathrm{II}}^2]\,.$$

L'équation (74) admet deux solutions en $Y = c^2/u^2$ qui sont définies par:

$$(75)\qquad \left.\begin{matrix} Y_1 \\ Y_2 \end{matrix}\right\} = \\ = \frac{(\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}})(2\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}} \sin^2\varphi) + \varepsilon_{\mathrm{II}}^2 \sin^2\varphi \pm \sqrt{[\varepsilon_{\mathrm{III}}(\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}}) + \varepsilon_{\mathrm{II}}^2]^2 \sin^4\varphi - 4\varepsilon_{\mathrm{I}}^2\varepsilon_{\mathrm{II}}^2 \cos^2\varphi}}{2(\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}} \sin^2\varphi)}\,.$$

Le système (72) a donc deux solutions correspondant aux signes + et — du 2ème membre de (75), ce qui met en évidence la biréfringence du plasma. (Remarquons qu'à chaque valeur de (75) correspondent deux solutions en c/u, mais ces deux déterminations impliquent seulement l'inversion du sens de la propagation le long de N; nous prendrons toujours dans la suite le signe +).

Les quantités ε_{I}, $\varepsilon_{\mathrm{II}}$ et $\varepsilon_{\mathrm{III}}$ sont déterminées par les six intégrales G qui sont des fonctions de ω, une fois ω_H fixée. L'équation (74) est donc une relation de la forme:

$$(74')\qquad \mathcal{F}\left(\omega, \frac{c}{u}\right) = 0\,,$$

qui représente l'équation de dispersion du milieu considéré.

La grandeur c/u représente l'indice de réfraction complexe et l'on détermine l'indice réel n et le facteur d'absorption k' en posant:

$$Y = \frac{c^2}{u^2} = (n - ik')^2 = L - iM , \tag{76}$$

d'où l'on tire:

$$\left| \begin{array}{l} n = \dfrac{1}{\sqrt{2}} \sqrt{L + \sqrt{L^2 + M^2}} , \\ k' = \dfrac{1}{\sqrt{2}} \sqrt{-L + \sqrt{L^2 + M^2}} . \end{array} \right. \tag{77}$$

et la vitesse de phase V et le coefficient d'atténuation β sont donnés par:

$$V = \frac{c}{n} , \qquad \beta = \frac{\omega k'}{c} . \tag{78}$$

Ces grandeurs se calculent donc directement en fonction des 6 intégrales G par les formules (70).

La vitesse de groupe des ondes W se calculera de même en utilisant la relation bien connue:

$$\frac{1}{W} = \frac{\mathrm{d}(n\omega)}{c\,\mathrm{d}\omega} = \frac{1}{cn}\left[n^2 + \frac{\omega}{2} \frac{\mathrm{d}(n^2)}{\mathrm{d}\omega} \right] . \tag{79}$$

Un cas particulier des équations (77), important pour les applications, est celui où $M \ll L$, car on peut alors écrire (si $L > 0$):

$$n \cong \sqrt{L} , \qquad k' \cong \frac{M}{2\sqrt{L}} . \tag{77'}$$

Nous allons étudier les deux cas particuliers bien connus de la propagation le long du champ magnétique et perpendiculaire à ce champ.

a) *Propagation longitudinale.* – On a dans ce cas $\varphi = 0$, d'où les deux solutions:

$$Y_{ol} = (\varepsilon_I + \varepsilon_{III}) + i\varepsilon_{II} , \qquad Y_{el} = (\varepsilon_I + \varepsilon_{III}) - i\varepsilon_{II} . \tag{80}$$

Nous verrons plus loin que Y_{ol} représente l'onde ordinaire et Y_{el} l'onde extraordinaire de la théorie habituelle, avec $\omega_H < 0$.

b) Propagation transversale. – On a alors $\varphi = \pi/2$, $\boldsymbol{H}$ est perpendiculaire à $\boldsymbol{N}$, d'où:

$$Y_{ot} = \varepsilon_{I}\,, \qquad Y_{et} = \frac{(\varepsilon_{I} + \varepsilon_{III})^2 + \varepsilon_{II}^2}{\varepsilon_{I} + \varepsilon_{III}}\,, \tag{81}$$

où Y_{ot} est le rayon ordinaire et Y_{et} le rayon extraordinaire.

10. – Polarisation des ondes.

Pout étudier la polarisation des ondes, il suffit de résoudre les équations (72) pour les valeurs de Y données par (75) et d'évaluer le rapport des composantes du champ électrique dans un plan perpendiculaire à $\boldsymbol{N}$, c'est-à-dire, avec nos notations, de calculer E_y/E_x. Remarquons d'ailleurs que nous avons dans notre système d'axes:

$$\left|\begin{array}{l} D_x = \dfrac{c}{u} H_y = \dfrac{c^2}{u^2} E_x\,, \\ D_y = -\dfrac{c}{u} H_x = \dfrac{c^2}{u^2} E_y\,, \\ D_z = H_z = 0\,, \end{array}\right. \tag{82}$$

de sorte que le rapport E_y/E_x peut aussi s'écrire:

$$\frac{E_y}{E_x} = \frac{D_y}{D_x} = -\frac{H_x}{H_y}\,. \tag{83}$$

D'après la 2ème équation du système (72), il vient:

$$\frac{E_y}{E_x} = \frac{(\varepsilon'_{yz}\varepsilon'_{xz} - \varepsilon'_{xy}\varepsilon'_{zz})}{\left(\dfrac{c^2}{u^2} - \varepsilon'_{yy}\right)\varepsilon'_{zz} + \varepsilon'^{2}_{yz}} = \frac{\varepsilon_{I}\varepsilon_{II}\cos\varphi}{\varepsilon_{I}(\varepsilon_{I} + \varepsilon_{III}) - (\varepsilon_{I} + \varepsilon_{III}\sin^2\varphi)\dfrac{c^2}{u^2}}\,. \tag{84}$$

Le second membre de (84) est en général un membre complexe qui détermine d'une part le rapport des modules de E_y et E_x et, d'autre part leur différence de phase; l'extremité du vecteur $\boldsymbol{D}$ (où ancore la projection de $\boldsymbol{E}$ sur xOy) décrit donc une ellipse dont le sens de parcours est donné par le signe du déphasage entre E_y et E_x.

Naturellement à chaque direction de propagation correspondent deux valeurs du rapport E_y/E_x, données par Y_1 et Y_2: pour chacune de ces valeurs

nous avons une ellipse dans le cas général et il est facile de démontrer que celles-ci sont parcourues en sens inverse. Posons en effet:

$$\left\{\begin{array}{l}\left(\dfrac{E_y}{E_x}\right)_{Y_1} = \dfrac{\varepsilon_{\mathrm{I}}\varepsilon_{\mathrm{II}}\cos\varphi}{\varepsilon_{\mathrm{I}}(\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}}) - (\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}}\sin\varphi)Y_1}\,, \\ \left(\dfrac{E_y}{E_x}\right)_{Y_2} = \dfrac{\varepsilon_{\mathrm{I}}\varepsilon_{\mathrm{II}}\cos\varphi}{\varepsilon_{\mathrm{I}}(\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}}) - (\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}}\sin^2\varphi)Y_2}\,.\end{array}\right.$$

En multipliant membre à membre ces deux équations, il vient:

$$\left(\frac{E_y}{E_x}\right)_{Y_1}\left(\frac{E_y}{E_x}\right)_{Y_2} =$$

$$= \frac{\varepsilon_{\mathrm{I}}^2\varepsilon_{\mathrm{II}}^2\cos^2\varphi}{\varepsilon_{\mathrm{I}}^2(\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}})^2 - \varepsilon_{\mathrm{I}}(\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}})(\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}}\sin^2\varphi)(Y_1+Y_2) + (\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}}\sin^2\varphi)^2 Y_1 Y_2}\,,$$

et, comme d'après (74):

$$Y_1 + Y_2 = \frac{(\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}})(2\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}}\sin^2\varphi) + \varepsilon_{\mathrm{II}}^2\sin^2\varphi}{\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}}\sin^2\varphi}\,,$$

$$Y_1 Y_2 = \frac{\varepsilon_{\mathrm{I}}[(\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}})^2+\varepsilon_{\mathrm{II}}^2]}{\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{III}}\sin^2\varphi}\,,$$

il vient:

$$\left(\frac{E_y}{E_x}\right)_{Y_1}\left(\frac{E_y}{E_x}\right)_{Y_2} = 1\,. \tag{85}$$

Les différences de phase entre E_y et E_x correspondant aux deux rapports sont donc de signes contraires et de sens de parcours inverses.

Nous appliquerons maintenant ces résultats aux cas particuliers des propagations longitudinale et transversale.

a) *Propagation longitudinale.* – On vérifie immédiatement, en utilisant (80) et (84) et en posant $\varphi = 0$, que:

$$\left(\frac{E_y}{E_x}\right)_{Y_{\mathrm{ol}}} = i\,, \qquad \left(\frac{E_y}{E_x}\right)_{Y_{\mathrm{el}}} = -i\,. \tag{86}$$

On obtient également à partir de (72), $E_z = 0$, de sorte que le champ électrique $\boldsymbol{E}$ est transversal. D'après (86), nous voyons que les ondes sont dans ce cas, polarisées circulairement: à Y_{ol} correspond une onde sinistrorsum

et l'on retrouve bien le rayon ordinaire de la théorie habituelle ([12-13]) (nous verifierons aussi au prochain chapitre que l'indice de réfraction, moyennant certaines approximations, est donné par les formules classiques); on a de même pour Y_{el} une onde dextrorsum qui définit le rayon extraordinaire.

b) Propagation transversale. – On trouve à partir de (81) et (84) les relations suivantes pour $\varphi = \pi/2$:

$$\left(\frac{E_y}{E_x}\right)_{Y_{et}} = \left(\frac{E_y}{E_x}\right)_{Y_{ot}} = 0 \ , \tag{87}$$

qui nous montrent que, pour les deux indices, les ondes sont linéairement polarisées.

A Y_{et} correspond le rayon extraordinaire; d'après (72) et (87), on voit tout de suite que $E_y = 0$ et que E_x et E_z sont $\neq 0$; le champ électrique $\boldsymbol{E}$ est donc perpendiculaire au champ magnétique constant $\boldsymbol{H}_0$ et sa composante dans le plan d'onde est dirigée selon Ox; mais il a aussi une composante longitudinale donnée par

$$E_z = \frac{\varepsilon'_{xz}}{\varepsilon'_{zz}} E_x = \frac{\varepsilon_{II}}{\varepsilon_I + \varepsilon_{III}} E_x \ .$$

Il en résulte que le vecteur électrique décrit une ellipse dans le plan xOz perpendiculaire à $\boldsymbol{H}_0$. Pour Y_{ot} nous avons immédiatement $E_x = E_z = 0$, le champ électrique est dirigé selon Oy; il est donc transversal et de plus parallèle au champ magnétique $\boldsymbol{H}_0$; c'est le rayon ordinaire qui se propage comme si $\boldsymbol{H}_0$ n'existait pas.

11. – Cas de l'ionosphère; fréquences limites et ondes courtes.

En vue des applications à la propagation des ondes dans l'ionosphère nous allons effectuer sur les 6 intégrales G certaines approximations dont nous discuterons la legitimité. Nous montrerons qu'on peut alors retrouver les résultats habituels, notamment pour les fréquences critiques, ce qui établira en même temps leur domaine de validité.

([12]) M. Mimno: *Rev. Mod. Phys.*, **9**, 1 (1937); R. Jouaust: *L'ionosphère* (Paris, 1946).

([13]) K. W. Wagner: *Lehre von den Schwingungen und Wellen* (1947); K. Rawer: *Die Ionosphäre* (Hollande, 1953).

Nous admettrons d'abord dans tous les calculs qui vont suivre que le terme en Γ_2^2 est négligeable (plus précisement que $\Gamma_2^2 C_\varphi m_1 \lambda / 6ACkT \ll 1$), ce qui revient à supposer que le champ électrique de l'onde incidente $\boldsymbol{E}$ est toujours faible: ceci étant toujours le cas pour les ondes radioélectriques se propageant dans l'ionosphère, nous n'aurons pas ici à nous occuper des corrections en Γ_2^2. Il en résulte comme nous l'avons vu (formule (41)) que l'approximation maxwellienne est valable.

La deuxième approximation consiste à negliger v_2^2/λ^2 devant ω^2, soit à écrire $v_2^2/\lambda^2 \ll \omega^2$. Cette approximation n'est pas rigoureusement valable puisque le rapport v_2/λ dépend en général de v_2 et que la distribution des vitesses des électrons permet d'attribuer à v_2 des valeurs quelconques; il nous faut donc supposer que l'on peut, sans erreur appréciable, remplacer v_2/λ par sa valeur moyenne, c'est-à-dire par la fréquence de collision $\nu = \bar{v}_2/\lambda$. Ceci étant admis, l'hypothèse que nous allons faire revient à supposer que la fréquence des collisions (entre électrons et ions ou molécules) est négligeable devant la fréquence des ondes, soit

$$\nu \ll \omega \,. \tag{88}$$

Dans le cas de l'ionosphère, on peut justifier cette approximation pour les couches F où la fréquence des chocs ([12] et [13]) varie entre 10 et 10^3 (chocs/s) et pour la couche E (10^4 à 10^5).

Par contre, elle n'est plus valable pour la couche D, où cette fréquence atteint 10^6 à 10^8 et elle n'est plus négligeable même pour les ondes courtes. Dans ce cas une étude rigoureuse des intégrales G devient nécessaire et l'on remarquera que leur valeur exacte dépend essentiellement de la loi d'interaction entre électrons et particules lourdes par l'intermédiaire de la grandeur $\lambda(v_2)$. Calculons maintenant les intégrales G avec l'hypothèse (88). Nous avons d'après la 1ère hypothèse (et d'après (39)) $U \cong 6ACkT$ et, d'après la seconde (et d'après (20), (21), (22) et (23)):

$$\left\{\begin{array}{l} z \cong \dfrac{\omega_H^2}{\omega^2}\,, \qquad A \cong \dfrac{\lambda\omega^2(1-z)^2}{1+z}\,, \qquad B \cong \dfrac{\lambda^2\omega^2(3-z)}{1+z}\,, \\ C \cong \dfrac{1+z}{(1-z)^2}\,, \qquad B' \cong \lambda^2\omega^2(1-z)\,. \end{array}\right. \tag{89}$$

En portant ces expressions dans (43), (44) et (45) on a successivement:

$$G_1 = \frac{1}{n_2}\int_0^\infty \frac{v_0^5 f_2^{(0)}\,\mathrm{d}v_2}{U} \cong \frac{1}{6kT\lambda\omega^2 n_2}\int_0^\infty f_2^{(0)} v_2^5\,\mathrm{d}v_2 = \tag{90}$$

$$= \frac{\overline{v_2^3}}{24\pi kT\lambda\omega^2} = \frac{1}{3\pi m_2\lambda\omega^2}\left(\frac{2kT}{\pi m_2}\right)^{\frac{1}{2}},$$

$$(91)\qquad G_1' = \frac{1}{n_2}\int_0^\infty C\lambda\,\frac{[(\lambda^2\omega^2+v_2^2)(1-z)-zB']v_2^4 f_2^{(0)}}{(\lambda^2\omega^2+v_2^2)(1+z)\,U}\,\mathrm{d}v_2 \simeq \frac{\overline{v_2^2}}{24\pi kT\omega^2} = \frac{1}{8\pi m_2\omega^2}\,,$$

$$(92)\qquad G_2 = \frac{1}{n_2}\int_0^\infty \frac{\lambda C v_2^4}{(\lambda^2\omega^2+v_2^2)\,U}\left[\lambda^2\omega^2\left(\frac{1-z}{1+z}\right) - v_2^2\right] f_2^{(0)}\mathrm{d}v_2 \cong$$

$$\cong \frac{\overline{v_2^2}}{24\pi kT\omega^2(1-z)} = \frac{1}{8\pi m_2(\omega^2-\omega_H^2)}\,,$$

$$(93)\qquad G_2' = \frac{1}{n_2}\int_0^\infty \frac{\lambda^2 C v_2^5 f_2^{(0)}\,\mathrm{d}v_2}{(\lambda^2\omega^2+v_2^2)(1+z)\,U} \cong \frac{\overline{v_2^3}}{24\pi kT\lambda(\omega^2-\omega_H^2)} =$$

$$= \frac{1}{3\pi m_2\lambda(\omega^2-\omega_H^2)^2}\left(\frac{2kT}{\pi m_2}\right)^{\frac{1}{2}}\,,$$

$$(94)\qquad G_3 = \frac{1}{n_2}\int_0^\infty \frac{zBv_2^5 f_2^{(0)}\,\mathrm{d}v_2}{(\lambda^2\omega^2+v_2^2-zB)\,U} \cong \frac{\omega_H^2(3\omega^2-\omega_H^2)\overline{v_2^3}}{24\pi kT\lambda\omega^2(\omega^2-\omega_H^2)^2} =$$

$$= \frac{\omega_H^2(3\omega^2-\omega_H^2)}{3\pi m_2\lambda\omega^2(\omega^2-\omega_H^2)^2}\left(\frac{2kT}{\pi m_2}\right)^{\frac{1}{2}}\,,$$

$$(95)\qquad G_3' = \frac{1}{n_2}\int_0^\infty \frac{CzB'\lambda v_2^4 f_2^{(0)}\,\mathrm{d}v_2}{(\lambda^2\omega^2+v_2^2)(1+z)\,U} \cong \frac{z\overline{v_2^2}}{24\pi kT\omega^2(1-z)} = \frac{\omega_H^2}{8\pi m_2\omega^2(\omega^2-\omega_H^2)}\,.$$

(Nous avons considéré dans ce calcul λ comme indépendant de v_2). Nous allons appliquer ces formules à différents cas particuliers.

1) *Cas où* $H_0 = 0$: - On a alors:

$$\omega_H = 0 \quad \text{et} \quad \varepsilon_{\mathrm{I}} = \varepsilon_{\mathrm{III}} = 0\,;$$

la biréfringence disparait et l'indice complexe est fourni dans tous les cas par:

$$\frac{c}{u} = \sqrt{\varepsilon_{\mathrm{I}}}\,.$$

D'après (70) et (76), on a pour les parties réelle et imaginaire, L et M, de c^2/u^2 (avec (90) et (91)):

$$L = 1 - 4\pi\mu G_1' = 1 - \frac{4\pi n_2 e_2^2}{m_2\omega^2} = 1 - \frac{\Omega^2}{\omega^2}\,,$$

$$M = 4\pi\mu\,\frac{G_1}{\omega} = \frac{32\pi n_2 e_2^2}{3m_2\lambda\omega^3}\left(\frac{2kT}{\pi m_2}\right)^{\frac{1}{2}} = \frac{8\Omega^2}{3\lambda\omega^3}\left(\frac{2kT}{\pi m_2}\right)^{\frac{1}{2}}\,.$$

où l'on a posé:

$$\Omega = \frac{4\pi n_2 e_2^2}{m_2}.$$

Si ω est suffisamment grand devant Ω, on voit immédiatement que $M \ll L$ et que les formules (77′) s'appliquent: l'indice de réfraction est donné par:

$$n = \sqrt{L} = \sqrt{1 - \frac{\Omega^2}{\omega^2}}, \tag{96}$$

et le coefficient d'absorption:

$$k' = \frac{M}{2\sqrt{L}} = \frac{\dfrac{4\Omega^2}{3\lambda\omega^3}\left(\dfrac{2kT}{\pi m_2}\right)^{\frac{1}{2}}}{\sqrt{1 - \dfrac{\Omega^2}{\omega^2}}}. \tag{97}$$

La fréquence limite est la valeur de ω qui annule n, soit évidemment: $\omega_c = \Omega$; on constate d'ailleurs que pour cette valeur le coefficient d'absorption devient infini. Pour une fréquence voisine de la fréquence limite, l'application des formules (77′) n'est plus légitime et il faut utiliser les formules générales (77).

Remarquons encore que $\sqrt{T}$ figure au numérateur de k', de sorte que le coefficient est proportionnel à la racine carrée de la température absolue du gaz ionisé. Aux températures usuelles de l'ionosphère, ce terme est petit devant ω, de sorte que l'application de (77′) reste légitime.

2) *Cas où* $H_0 \neq 0$. – Nous appliquerons les résultats des formules (90) à (95) aux deux cas particuliers déjà étudiés.

a) *Propagation longitudinale.* – Nous avons pour le rayon ordinaire, d'après (70):

$$Y_{ol} = (\varepsilon_I + \varepsilon_{III}) + i\varepsilon_{II} = 1 - 4\pi\mu\left[(G_1' + G_3') + \frac{\omega_H}{\omega} G_2\right] -$$

$$- i4\pi\mu\left[\frac{(G_1 + G_3)}{\omega} + 2\omega_H G_2'\right],$$

d'où en utilisant les formules (90) à (95):

$$L_{ol} = 1 - \frac{\Omega^2}{\omega^2}\left(1 + \frac{\omega_H^2}{\omega^2 - \omega_H^2}\right) - \frac{\omega_H}{\omega}\frac{\Omega^2}{(\omega^2 - \omega_H^2)} = \tag{98}$$

$$= 1 - \frac{\Omega^2}{\omega^2 - \omega_H^2}\left(1 + \frac{\omega_H}{\omega}\right) = 1 - \frac{\Omega^2}{\omega(\omega - \omega_H)},$$

$$M_{o1} = \frac{8\Omega^2}{3\lambda\omega^3}\left(\frac{2kT}{\pi m_2}\right)^{\frac{1}{2}}\left[1 + \frac{\omega_H^2(3\omega^2 - \omega_H^2)}{(\omega^2 - \omega_H^2)^2} + \frac{2\omega^3\omega_H}{(\omega^2 - \omega_H^2)^2}\right] = \tag{99}$$

$$= \frac{8\Omega^2}{3\lambda\omega(\omega - \omega_H)^2}\left(\frac{2kT}{\pi m_2}\right)^{\frac{1}{2}}.$$

Si ω est suffisamment grand devant la gyrofréquence, on voit facilement que $M_{o1} \ll L_{o1}$, de sorte que l'indice de réfraction est donné par:

$$n_{o1} = \sqrt{1 - \frac{\Omega^2}{\omega(\omega - \omega_H)}}, \tag{100}$$

et l'absorption par:

$$k'_{o1} = \frac{M_{o1}}{2\sqrt{L_{o1}}}.$$

On a de même pour le rayon extraordinaire:

$$Y_{e1} = (\varepsilon_I + \varepsilon_{III}) - i\varepsilon_{II} = 1 - 4\pi\mu\left[(G'_1 + G'_3) - \frac{\omega_H}{\omega}G_2\right] -$$

$$- i4\pi\mu\left[\frac{(G_1 + G_3)}{\omega} - 2\omega_H G'_2\right],$$

d'où:

$$L_{e1} = 1 - \frac{\Omega^2}{\omega(\omega + \omega_H)}, \tag{101}$$

$$M_{e1} = \frac{8\Omega^2}{3\lambda\omega(\omega + \omega_H)^2}\left(\frac{2kT}{\pi m_2}\right)^{\frac{1}{2}}. \tag{102}$$

Si l'on peut appliquer l'approximation (77′), on a:

$$n_{e1} = \sqrt{1 - \frac{\Omega^2}{\omega(\omega + \omega_H)}}, \tag{103}$$

et

$$k'_{e1} = \frac{M_{e1}}{2\sqrt{L_{e1}}}.$$

Les formules (100) et (103) correspondent bien aux formules habituelles donnant l'indice de refraction des rayons ordinaire et extraordinaire pour la

propagation longitudinale. En particulier les fréquences limites sont:

$$(104)\qquad \begin{cases} \omega_{o1}^{c} = \dfrac{\omega_H}{2} + \sqrt{\Omega^2 + \dfrac{\omega_H^2}{4}}\ , \\ \omega_{e1}^{c} = -\dfrac{\omega_H}{2} + \sqrt{\Omega^2 - \dfrac{\omega_H^2}{4}}\ . \end{cases}$$

Les mêmes remarques que celles du 1er sont valables pour le coefficient d'absorption, la validité des formules (77′) et la présence du terme en $\sqrt{T}$ dans k'.

b) Propagation transversale. – D'après les formules (81), le rayon ordinaire est donné par:

$$Y_{ot} = \varepsilon_{\mathrm{I}}\ ,$$

de sorte que

$$L_{ot} = 1 - \frac{\Omega^2}{\omega^2}\ ; \qquad M_{ot} = \frac{8\Omega^2}{3\lambda\omega^3}\left(\frac{2kT}{\pi m_2}\right)^{\frac{1}{2}}\ ,$$

de sorte que l'onde ordinaire se propage comme s'il n'y avait pas de champ magnétique (cas du 1er); on a, avec les mêmes réserves que précédemment:

$$(105)\qquad n_{ot} = \sqrt{1 - \frac{\Omega^2}{\omega^2}}\ .$$

Le rayon extraordinaire est donné par:

$$Y_{et} = \frac{(\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}})^2 + \varepsilon_{\mathrm{II}}^2}{\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}}}\ .$$

Nous calculerons seulement la partie réelle de cette expression, soit:

$$L_{et} = 1 - 4\pi\mu(G_1' + G_3') + \frac{[1 - 4\pi\mu(G_1' + G_3')]\cdot 16\pi^2\mu^2\omega_H^2\left(4G_2'^2 - \dfrac{G_2^2}{\omega^2}\right)}{[1 - 4\pi\mu(G_1' + G_3')]^2 + 16\pi^2\mu^2\dfrac{(G_1 + G_3)^2}{\omega^2}}\ ,$$

termes en

$$\frac{(G_1 + G_3)G_2G_2'}{\omega^2|\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}}|^2}\ .$$

Avec les hypothèses faites précédemment, on peut voir aisément que le

3ème terme de l'expression précédente est négligeable, ainsi que les termes en $G_2'^2$ et $(G_1+G_3)^2/\omega^2$. En appliquant les formules (90) à (95), il vient:

$$L_{et}=1-\frac{\Omega^2}{\omega^2-\omega_H^2}-\frac{\dfrac{\omega_H^2\Omega^2}{\omega^2(\omega^2-\omega_H^2)^2}}{1-\dfrac{\Omega^2}{\omega^2-\omega_H^2}}=1-\frac{\Omega^2(\omega^2-\Omega^2)}{\omega^2(\omega^2-\omega_H^2-\Omega^2)}\,.$$

On peut aussi montrer que pour ω assez grand, $M_{et}\ll L_{et}$, de sorte que l'indice de réfraction du rayon extraordinaire est

$$n_{et}=\sqrt{1-\frac{\Omega^2(\omega^2-\Omega^2)}{\omega^2(\omega^2-\omega_H^2-\Omega^2)}}\,, \tag{106}$$

ce qui correspond bien à l'expression habituelle.

3) *Approximations quasi-longitudinale et quasi-transversale.* – Dans le cas général, on doit appliquer la formule (75); mais on peut montrer, en nous plaçant dans les approximations précédentes, que, selon l'importance relative des termes sous le radical de (75), on a une propagation de type longitudinal ou de type transversal. Nous aurons ainsi l'approximation quasi-longitudinale (Q.L.) si:

$$|\varepsilon_{III}(\varepsilon_I+\varepsilon_{III})+\varepsilon_{II}^2|\sin\varphi\,\mathrm{tg}\,\varphi\ll 2\,|\varepsilon_I|\cdot|\varepsilon_{II}|$$

où l'approximation quasi-transversale (Q.T.) si:

$$|\varepsilon_{III}(\varepsilon_I+\varepsilon_{III})+\varepsilon_{II}^2|\sin\varphi\,\mathrm{tg}\,\varphi\gg 2\,|\varepsilon_I|\cdot|\varepsilon_{II}|\,.$$

Avec les approximations précédentes, ces deux conditions s'écrivent:

$$\frac{1}{2}\,|\omega_H|\sin\varphi\,\mathrm{tg}\,\varphi\ \lessgtr\ \frac{\omega^2-\Omega^2}{\omega} \tag{107}$$

et nous retrouvons ainsi des résultats connus ([13]).

La condition (107) dépend étroitement de la densité électronique Ω^2. Pour des densités faibles, c'est l'approximation (Q.L.) qui est valable (le voisinage de l'équateur magnétique mis à part, car φ est alors voisin de $\pi/2$), pour des densités élevées c'est l'approximation (Q.T.). En général c'est presque toujours l'approximation (Q.T.) qui est valable au voisinage du point de réflexion; des exceptions se présentent seulement pour une propagation presque longitudinale (au voisinage des pôles magnétiques pour l'ionosphère) où l'on observe aussi des réflexions du type longitudinal. On obtient alors 3 rayons réfléchis, le

rayon ordinaire de type longitudinal étant appelé le rayon « Z » (voir fig. 1). Il résulte d'un couplage des composante ordinaire et extraordinaire que nous avons étudié en détail dans un autre travail [14].

Nous remarquerons encore que, d'après la formule générale (75) Y a un pôle pour une valeur de ω telle que:

$$\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{III}} \sin^2 \varphi = 0$$

(avec le signe — au numérateur) qui pourrait éventuellement fournir une réflexion.

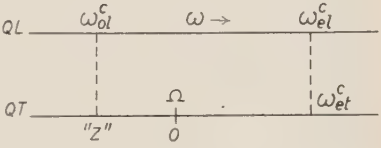

Fig. 1.

Si nous prenons les approximations précédentes, la relation ci-dessus est vérifiée pour $\omega = \omega_r^c$ avec:

$$\Omega^2 = \frac{\omega_r^{c2}}{1 - \dfrac{\omega_H^2 \sin^2 \varphi}{\omega_r^{c2} - \omega_H^2}}, \tag{108}$$

ce qui correspond au rayon signalé par Rai [15]; nous avons d'ailleurs montré. que le rayon a très peu de chances d'être observé [14].

4) *Ondes courtes.* - On obtient immédiatement les formules valables pour les ondes courtes en développant les formules (100), (103) et (106) par rapport à ω_H/ω et en négligeant les termes en ω_H^2/ω^2, d'où l'on tire les formules connues:

$$n'_{o1} = \sqrt{1 - \frac{\Omega^2}{\omega^2}\left(1 + \frac{\omega_H}{\omega}\right)}, \tag{100'}$$

$$n'_{e1} = \sqrt{1 - \frac{\Omega^2}{\omega^2}\left(1 - \frac{\omega_H}{\omega}\right)}, \tag{103'}$$

$$n'_{ot} = n'_{et} = \sqrt{1 - \frac{\Omega^2}{\omega^2}}. \tag{106'}$$

12. – Conclusion.

L'étude précedente nous a permis d'établir les propriétés des plasmas faiblement ionisés en nous basant sur l'approximation du 1er ordre de l'équation

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[14] R. Jancel et T. Kahan: *Journ. Phys. et le Rad.*, sous presse.

[15] T. L. Eckersley: *Proc. Phys. Soc.*, B **361**, 49 (1950); R. N. Rai: *Proc., Nat. Inst. Sci.*, **3**, 307 (1937); O. E. H. Rydbeck: *Trans. Chalmers Univ. of Techn. Gothenburg*, 101 (1951).

de Boltzmann. Cette méthode présente l'avantage d'être utilisable pour l'étude des cas non maxwelliens et surtout de fournir les expressions exactes pour les diverses propriétés du gaz ionisé, en s'affranchissant des hypothèses peu justifiées qui limitent beaucoup la portée de la methode basée sur le libre parcours moyen.

Nous avons pu ainsi établir des formules générales qui comportent toutes, comme cas particuliers, les expressions élémentaires utilisées couramment (conductivité, formule de Langevin, tenseur diélectrique, coefficient de Hall); de plus un autre avantage de la méthode est de pouvoir préciser la nature exacte et le sens physique des approximations faites et d'en délimiter ainsi le domaine de validité. Nous avons ensuite appliqué ces résultats à la propagation des ondes électromagnétiques planes dans un tel milieu; nous avons déterminé les divers modes de propagation et établi une formule générale de dispersion qui est l'expression généralisée de la formule d'Appleton et de Hartree ([16]) et qui permet d'analiser rigoureusement la structure des ondes électromagnétiques transmises à partir des grandeurs G qui definissent le milieu.

Pour le cas de l'ionosphère, les formules classiques ont bien été retrouvées en faisant l'hypothèse que la fréquence des chocs est négligeable devant la fréquence des ondes; plus précisement nous avons vu que cette approximation n'était elle-même valable qu'à condition de négliger la distribution des vitesses pour le rapport r_2^2/λ^2. Il est légitime de supposer qu'un calcul rigoureux modifierait sensiblement les résultats quantitatifs sans en altérer l'interprétation qualitative globale, surtout pour les ondes courtes. Nous avons, par contre, remarqué que les approximations perdaient tout leur sens dans la couche D de l'ionosphère même pour les ondes courtes, ce qui offre un champ d'application intéressant. Notons encore qu'un calcul rigoureux des expressions G ferait intervenir la loi d'interaction entre électrons et molécules (ou atomes ionisés) par l'intermédiaire de $\lambda(v_2)$; enfin les formules précédentes sont évidemment applicables à des milieux ionisés différents où les conditions présentes dans l'ionosphère ne sont pas réalisées.

Les conclusions précédentes ne sont valables que pour une ionosphère homogène, ou localement dans une ionosphère non homogène; les conditions de réflexion et de réfraction dans ce dernier cas ont été étudiées dans un autre travail ([14]), où nous avons considéré les grandeurs ε_{I}, $\varepsilon_{\mathrm{II}}$ et $\varepsilon_{\mathrm{III}}$ comme des fonctions lentement variables de z. On retrouve ainsi les fréquences critiques ω_{ol}^c, Ω, ω_{et}^c ainsi que le rayon de Rai et l'on peut alors rendre compte du phénomène de triple décomposition ([14,15]).

Nous terminerons en remarquant que les calculs sur la propagation ionosphérique du § 11 reposent sur l'hypothèse fondamentale suivante: la distri-

([16]) E. V. Appleton: *Journ. Inst. Electr. Eng.*, **71**, 642 (1932); D. R. Hartree: *Proc. Phys. Soc.*, **27**, 143 (1931).

bution des vitesses électroniques est maxwellienne avec une « température électronique » égale à celle du gaz. En réalité ceci n'est probablement pas valable en toute rigueur pour les couches ionosphériques E et F, car il convient de tenir compte de l'action des chocs inélastiques et des phénomènes d'ionisation et de recombinaison; il en résulte que l'énergie moyenne des électrons peut être supérieure à celle des molécules ou atomes et la répartition n'être pas maxwellienne, bien que le champ $\boldsymbol{E}$ de l'onde radioélectrique soit négligeable: on peut même avoir une répartition maxwellienne mais avec une « température électronique » nettement supérieure à la temperature ambiante. Tous ces phénomènes peuvent intervenir pour modifier sensiblement les résultats du présent travail sur la propagation ionosphérique et nécessitent d'une étude approfondie de la structure fine de l'ionosphère.

Ajouté en épreuves.

Dans une lettre sous presse au *Journal de Physique et le Radium* nous avons montré que la formule de biréfringence ionosphérique établie par APPLETON [17] dans le cas où l'absorption est négligeable se déduit de nos formules générales en faisant les hypothèses approchées suivantes:

1) l'ionosphère biréfringente est assimilée à un plasma lorentzien;

2) la fonction de distribution des vitesses électroniques est maxwellienne (cas d'un champ électrique de faible intensité); ,

3) le nombre de choc ν est très petit par rapport à ω (la validité de cette hypothèse est d'ailleurs discutée dans le présent mémoire d'une manière approfondie). On vérifie en effet qu'en portant dans (75) qui définit l'indice de réfraction, les valeurs approchées de ε_{I}, $\varepsilon_{\mathrm{II}}$ et $\varepsilon_{\mathrm{III}}$ tirées de (90), (91), (92), (93) et (95) à l'aide des hypothèses ci-dessus

$$(109)\qquad \varepsilon_{\mathrm{I}} \cong 1 - \frac{\omega^2}{\Omega^2}, \qquad \varepsilon_{\mathrm{II}} \cong i\,\frac{\omega_H}{\omega}\,\frac{\Omega^2}{\omega^2 - \omega_H^2}, \qquad \varepsilon_{\mathrm{III}} \cong -\,\frac{\Omega^2\omega_H^2}{\omega^2(\omega^2 - \omega_H^2)},$$

on retrouve par un calcul direct mais laborieux, l'expression de l'indice n

$$(110)\qquad n^2 = 1 - \frac{\Omega^2}{\omega^2 - \frac{1}{2}\,\frac{\omega^2\omega_H^2}{\omega^2 - \Omega^2}\sin^2\varphi \pm \sqrt{\omega^2\omega_H^2\cos^2\varphi + \left[\frac{1}{2}\,\frac{\omega^2\omega_H^2}{\omega^2 - \Omega^2}\sin^2\varphi\right]^2}};$$

c'est précisément la formule d'Appleton sans terme de frottement phénoméno-

[17] APPLETON: *Journ. Inst. Electr. Eng. Wireless Section*, **7**, 257 (1932).

logique. Notre formule générale (75) permet également d'obtenir l'indice de réfraction complexe pour le cas où l'absorption est différente de zéro. Pour ce faire on est amené à tenir compte des trois intégrales fondamentales G_1, G_2' et G_3 qui traduisent l'atténuation due aux chocs, ce qui revient à supprimer la 3ème ipothèse ci-dessus. Elles permettent d'évaluer de manière correcte l'affaiblissement, donc l'absorption, et conduisent ainsi à restreindre la validité de la généralisation de la formule d'Appleton obtenue par l'introduction d'un terme phénoménologique de frottement, généralisation qui est loin de représenter les phénomènes observés. Ces calculs sont en cours et feront l'objet d'une publication ultérieure.

RIASSUNTO (*)

Appoggiandosi su un metodo per la risoluzione dell'equazione integro-differenziale di Boltzmann ricavato dagli autori [11], metodo che generalizza risultati precedenti [4], si calcola la funzione di ripartizione delle velocità elettroniche in un gas ionizzato anisotropo (sottoposto a un campo magnetico costante) per stati *non maxwelliani*. Questa permette di ottenere espressioni esplicite per la conduttività magneto-ottica ionica, per il tensore dielettrico, per l'effetto Hall, per la deviazione di un fascetto elettronico e per una generalizzazione della formula di mobilità di Langevin. Si fa un confronto di questi risultati con quelli forniti da altri metodi di calcolo (basati sul libero percorso medio). Si studia poi la propagazione delle onde elettromagnetiche piane in un plasma di tal genere; in particolare si stabiliscono formule per l'indice di rifrazione, la birifrangenza, la polarizzazione delle onde e le frequenze limite. Con determinate approssimazioni, delle quali si discute la validità, si ritrovano i risultati delle teorie classiche relative alla ionosfera (Appleton, Hartree, ecc.).

(*) *Traduzione a cura della Redazione.*

Magnetic Effects in the Scattering of Muons by Nuclei.

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Summary. — In connection with recent experiments on the scattering of μ-mesons by nuclei the effect of the anomalous magnetic moment of the incident muon and of the distribution of currents in the nucleus have been evaluated. The results are for non-polarized beams incident on non-polarized nuclei. At relatively large angles the cross-sections depend sensitively on the way in which the charges and currents are distributed in the nucleus.

Recent work [1] on the scattering of cosmic ray muons by nuclei seems to confirm the existence of an anomalous scattering [2]. In this note we shall briefly discuss what contributions to the cross-section may arise from an anomalous magnetic moment of the impinging muon, when taking into account the finite extension of the nucleus and the electric current density in its interior. Apart from the present difficulties the evaluation of more detailed effects in the scattering of muons by nuclei should become of interest when better experimental data will permit better comparisons.

To describe the nuclear charge distribution ϱ and the nuclear current distribution $\boldsymbol{j}$ a static approximation will be used, so that a solenoidal condition for the currents must be satisfied. Theoretically we think that $\boldsymbol{j}(\boldsymbol{r})$ is derived from the nuclear ground state wave function $\Psi(\boldsymbol{r}_1, \sigma_z^{(1)}; \boldsymbol{r}_2, \sigma_z^{(2)}; \ldots)$ for instance in the form $\boldsymbol{j}(\boldsymbol{r}) = c\nabla\wedge\big[\boldsymbol{M}_c(\boldsymbol{r}) + \boldsymbol{M}_s(\boldsymbol{r})\big]$, $\boldsymbol{M}_c(\boldsymbol{r})$ being a con-

[1] I. B. McDiarmid: *Phil. Mag.*, **45**, 933 (1954); G. D. Rochester and A. W. Wolfendale: *Phil. Mag.*, **45**, 980 (1954).

[2] W. Whittmore and R. P. Shutt: *Phys. Rev.*, **88**, 1312 (1952); E. P. George, J. L. Redding and P. T. Trent: *Proc. Phys. Soc.*, A **66**, 553 (1953); B. Leontic and A. W. Wolfendale: *Phil. Mag.*, **44**, 1101 (1953); M. L. T. Kannangara and G. S. Shrikantia: *Phil. Mag.*, **44**, 1091 (1953).

vection magnetic moment distribution and $\boldsymbol{M}_s(\boldsymbol{r})$ a spin magnetic moment distribution ([3]). To satisfy the solenoidal condition for $\boldsymbol{j}(\boldsymbol{r})$ we take the vector potential $\boldsymbol{A}(\boldsymbol{r})$ in the form $g\mu_B\nabla A_0(\boldsymbol{r})\wedge\boldsymbol{I}$, where g denotes the gyromagnetic ratio of the nucleus, μ_B is the nuclear Bohr magneton, $A_0(\boldsymbol{r})$ is to be determined from the assumed current density, and $\boldsymbol{I}$ denotes the spin of the nucleus. We observe that $A_0(\boldsymbol{r})$ must satisfy the asymptotic condition $A_0(\boldsymbol{r})\sim 1/|\boldsymbol{r}|$ for large $|\boldsymbol{r}|$ values. Moreover we shall assume that $A_0(\boldsymbol{r})$ depends on $\boldsymbol{r}$ only through $r=|\boldsymbol{r}|$. The current density $\boldsymbol{j}(\boldsymbol{r})$ is then given by

$$\boldsymbol{j}(\boldsymbol{r}) = -(c/4\pi)g\mu_B\nabla^2\big[\nabla A_0(\boldsymbol{r})\wedge\boldsymbol{I}\big] = j_0\varkappa(r)[\text{vers }\boldsymbol{I}\wedge\text{vers }\boldsymbol{r}]\,,$$

where $j_0\varkappa(r)$ is the current distribution along any radius perpendicular to $\boldsymbol{I}$. The solenoidal character of this last expression is clearly exhibited. We assume the appropriate normalization for $\varkappa(r)$, which as will next be shown, consists in requiring the overall space integral of $r\varkappa(r)$ to be unity. In principle j_0 is then fixed from the assumed asymptotic condition

$$j_0\frac{4\pi}{c}(g\mu_B I)^{-1}\int^{r}\mathrm{d}r'r'^{-2}\int^{r'}\mathrm{d}r''r''^{2}\int^{r''}\mathrm{d}r'''\varkappa(r''')\sim 1/r\,,$$

since the continuity of $A_0(r)$, at least, must be required. One sees, however, that this last relation does not immediately provide a simple formula for j_0. A more convenient procedure is the following. The asymptotic condition $A_0(\boldsymbol{r})\sim 1/r$ at infinity entails $\mathcal{F}\{A_0(\boldsymbol{r})\}\sim(4\pi/K^2)$ for $K\to 0$ ($\mathcal{F}\{f(\boldsymbol{r})\}=$ Fourier transform of $f(\boldsymbol{r})$). Consider the scalar product $j_0\mathcal{F}\{\varkappa(r)\,\text{vers }\boldsymbol{r}\}\cdot\boldsymbol{K}=(c/4\pi)\cdot\cdot g\mu_B I\mathcal{F}\{\nabla[\nabla^2A_0(\boldsymbol{r})]\}\cdot\boldsymbol{K}$. It can be put in the form $i(c/4\pi)g\mu_B IK^4\mathcal{F}\{A_0(\boldsymbol{r})\}$ after two partial integrations. (When doing this we shall always suppose a convergence factor to be present. However no special prescriptions are required for evaluating the final integrals). Comparing the two expressions it easily follows that

$$\mathcal{F}\{A_0(\boldsymbol{r})\}=j_0\frac{4\pi}{c}(g\mu_B I)^{-1}K^{-2}\left(\frac{\pi}{2}\right)^{-\frac{1}{2}}\int\mathrm{d}\boldsymbol{r}\,r\varkappa(r)(Kr)^{-\frac{3}{2}}J_{\frac{3}{2}}(Kr)\,,$$

($J_{\frac{3}{2}}$ denoting the Bessel function of order $\frac{3}{2}$). Now, passing to the limit, we readily get $j_0=3c(g\mu_B I)$ and finally (this formula will be used in the following)

$$\mathcal{F}\{A_0(\boldsymbol{r})\}=12\pi K^{-2}\left(\frac{\pi}{2}\right)^{-\frac{1}{2}}\int\mathrm{d}\boldsymbol{r}\,r\varkappa(r)(Kr)^{-\frac{3}{2}}J_{\frac{3}{2}}(Kr)\,. \tag{1}$$

([3]) See J. M. Blatt and V. F. Weisskopf: *Theoretical Nuclear Physics* (New York, 1952), Chap. 1-7-B.

For the limiting case of a point nucleus $\varkappa(r)$ is given by $-\frac{1}{3}$ vers $\boldsymbol{r}\cdot\nabla\delta(\boldsymbol{r})$ (the factor $\frac{1}{3}$ because of $\nabla\cdot\boldsymbol{r}=3$). Using well known properties of the δ-function and of its derivative it follows that $\mathcal{F}\{A_0(\boldsymbol{r})\} = (4\pi/K^2)$ everywhere. Thus, we see that, from a consistent use of the solenoidal condition for the currents, the current density is essentially determined by the form chosen for $\varkappa(r)$. The Fourier transform of $A_0(\boldsymbol{r})$, which occurs in the cross-sections, is also determined by $\varkappa(r)$ through relation (1).

The additional term in the interaction hamiltonian we assume to be [4]

$$\lambda\bar{\mu}\sigma_{\mu\nu}F_{\mu\nu}\,;$$

λ being a pure number. In non-covariant notations it may be written

$$-\lambda\bar{\mu}\big[\beta(\boldsymbol{\sigma}\cdot\nabla\wedge\boldsymbol{A}(\boldsymbol{r})) + i\beta(\boldsymbol{\alpha}\cdot\nabla V(\boldsymbol{r}))\big],$$

where $V(\boldsymbol{r}) = eV_0(\boldsymbol{r})$ is the electric potential. Two terms in the total interaction hamiltonian contain the vector potential $\boldsymbol{A}(\boldsymbol{r})$ and they may thus lead to a change in the nuclear polarization. We therefore consider (the Born approximation is used) transitions from an initial state $\Psi(m_{\text{in}})\,\mathfrak{u}_{\text{in}}\exp[i\boldsymbol{K}_{\text{in}}\cdot\boldsymbol{r}]$ to a final state $\Psi(m_{\text{fin}})\mathfrak{u}_{\text{fin}}\exp[i\boldsymbol{K}_{\text{fin}}\cdot\boldsymbol{r}]$, where $\Psi(m)$ is the nuclear ground state wave function pertaining to the eigenvalue m of I_z and $\mathfrak{u}_{\text{in}}$ and $\mathfrak{u}_{\text{fin}}$ are the usual spinor quantities of the initial and final state respectively. To obtain the cross-section for unpolarized incident beam and for unpolarized nuclei we must average over initial polarizations (of the nucleus and of the particle) and sum over the final polarizations (of the nucleus and of the particle). To avoid confusion, since a product space intervenes, we will reserve in the following the name *spur* to the usual diagonal sums on spinor indices, the name *trace* to diagonal sums on nuclear polarization indices (m_{in}, m_{fin}), and the prefix *diag. sum* when summing on 3-dimensional space indices (r, s, t, u, v, w). For an unpolarized nucleus no interference terms may arise between the electric (i.e. containing the electric potential) and the magnetic part (i.e. containing the vector potential) of the matrix element. This may be seen as follows. Such an interference term will contain linearly the nuclear spin $\boldsymbol{I}$, so that, when summing over the nuclear polarizations, a factor *trace* $\boldsymbol{I}$ will appear. But *trace* $\boldsymbol{I}=0$, as immediately follows from the commutation rules. The electric part of the matrix element may be put, after a partial integration, in the form

$$\mathcal{E} = -\mathcal{F}\{V(\boldsymbol{r})\}\big[e(\mathfrak{u}_{\text{in}}^*|\,\mathfrak{u}_{\text{fin}}) + \bar{\mu}(\mathfrak{u}_{\text{in}}^*|\beta(\boldsymbol{\alpha}\cdot\boldsymbol{K})|\,\mathfrak{u}_{\text{fin}})\big],$$

supposing that the selection rule $m_{\text{in}} = m_{\text{fin}}$, is satisfied ($\boldsymbol{K}$, which also appears in the Fourier transform is given by $\boldsymbol{K} = \boldsymbol{K}_{\text{fin}} - \boldsymbol{K}_{\text{in}}$). The magnetic part of

[4] W. Pauli: *Die Allgemeinen Prinzipien der Wellenmechanik*, in *Handbuch der Physik*, **24/1** (Berlin, 1933), p. 233.

the matrix element after successive partial integrations may be given the form

$$\mathcal{M} = g\bar{\mu}\,\mathcal{F}\{A_0(\boldsymbol{r})\}\big[ie(\mathrm{u}_{\mathrm{in}}^*|\boldsymbol{\alpha}|\mathrm{u}_{\mathrm{fin}})\cdot[\boldsymbol{K}\wedge(m_{\mathrm{in}}|\boldsymbol{I}|m_{\mathrm{fin}})] + \\ + \bar{\mu}\,(\mathrm{u}_{\mathrm{in}}^*|\beta\boldsymbol{\sigma}|\mathrm{u}_{\mathrm{fin}})[\boldsymbol{K}\wedge[\boldsymbol{K}\wedge(m_{\mathrm{in}}|\boldsymbol{I}|m_{\mathrm{fin}})]]\big].$$

To reduce to α-matrices only we substitute:

$$\boldsymbol{\sigma}\cdot[\boldsymbol{K}\wedge[\boldsymbol{K}\wedge(m_{\mathrm{in}}|\boldsymbol{I}|m_{\mathrm{fin}})]] = -\,i(\boldsymbol{\alpha}\cdot\boldsymbol{K})[\boldsymbol{\alpha}\cdot[\boldsymbol{K}\wedge(m_{\mathrm{in}}|\boldsymbol{I}|m_{\mathrm{fin}})]].$$

Next we introduce a matrix $\mathbf{T}$ whose matrix elements are

$$\mathbf{T}_{rs} = e^2(\mathrm{u}_{\mathrm{in}}^*|\alpha_r|\mathrm{u}_{\mathrm{fin}})(\mathrm{u}_{\mathrm{fin}}^*|\alpha_s|\mathrm{u}_{\mathrm{in}}) - e\mu(\mathrm{u}_{\mathrm{in}}^*|\alpha_r|\mathrm{u}_{\mathrm{fin}})(\mathrm{u}_{\mathrm{fin}}^*|\alpha_s(\boldsymbol{\alpha}\cdot\boldsymbol{K})\beta|\mathrm{u}_{\mathrm{in}}) - \\ - e\mu(\mathrm{u}_{\mathrm{in}}^*\,\beta(\boldsymbol{\alpha}\cdot\boldsymbol{K})\alpha_r\,\mathrm{u}_{\mathrm{fin}})(\mathrm{u}_{\mathrm{fin}}^*\,\alpha_s\,\mathrm{u}_{\mathrm{in}}) + \mu^2(\mathrm{u}_{\mathrm{in}}^*\,\beta(\boldsymbol{\alpha}\cdot\boldsymbol{K})\alpha_r\,\mathrm{u}_{\mathrm{fin}})(\mathrm{u}_{\mathrm{fin}}^*|\alpha_s(\boldsymbol{\alpha}\cdot\boldsymbol{K})\beta\,|\,\mathrm{u}_{\mathrm{in}}),$$

and we choose to sum first over the nuclear polarizations. We adopt the usual sum convention for spatial indices and for convenience we use the Levi-Civita symbol δ_{ijk}. It follows that

$$\frac{1}{2I+1}\sum_{m_{\mathrm{in}}}\sum_{m_{\mathrm{fin}}}|\mathcal{M}|^2 = \\ = \frac{g^2\bar{\mu}^2}{2I+1}\,\mathcal{F}\{A_0(\boldsymbol{r})\}\,\mathbf{T}_{rs}\delta_{rtu}\delta_{svw}K_tK_v\sum_{m_{\mathrm{in}}}\sum_{m_{\mathrm{fin}}}(m_{\mathrm{in}}|I_u|m_{\mathrm{fin}})(m_{\mathrm{fin}}|I_w|m_{\mathrm{in}}).$$

The double sum reduces to a *trace* due to the fact that $\Psi(m)$ form a closed set. To evaluate it is now very simple, for one can reduce to the z-axis by similarity transforms. The above expression then becomes

$$g^2\bar{\mu}^2|\mathcal{F}\{A_0(\boldsymbol{r})\}|^2[\delta_{sr}\delta_{vt} - \delta_{tr}\delta_{st}]K_tK_v\mathbf{T}_{rs}\tfrac{1}{3}I(I+1) = \\ = g^2\bar{\mu}^2|\mathcal{F}\{A_0(\boldsymbol{r})\}|^2\tfrac{1}{3}I(I+1)[-\mathbf{T}_{rs}K_rK_s + K^2\ \widehat{diag\text{-}sum}\ \mathbf{T}].$$

The remaining part of the calculation involves only ordinary *spur* technique. We shall not refer it since the procedure is quite straightforward though somewhat lengthy. The final result we put in the form

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right) = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)_{R^*}[1 + f_{\mathrm{electric}} + f_{\mathrm{magnetic}}],$$

where $(\mathrm{d}\sigma/\mathrm{d}\omega)_{R^*}$ is the Rutherford relativistic cross-section for the finite nucleus. The two factors f_{electric} and f_{magnetic} are given by

$$f_{\mathrm{electric}} = \sin^2\frac{\theta}{2}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)^{-1}\beta^2(1-\beta^2)[2\lambda(1-\beta^2)+\lambda^2\beta^2], \tag{2}$$

$$(3)\qquad f_{\text{magnetic}} = \frac{2}{3}\left(\frac{m_0}{M}\right)^2 (gI)^2 \frac{I-1}{I}\,\beta^4(1-\beta^2)^{-1}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)^{-1}\left|\frac{G(\boldsymbol{p}-\boldsymbol{p}_0)}{F(\boldsymbol{p}-\boldsymbol{p}_0)}\right|^2 \sin^{-4}\frac{\theta}{2}\cdot$$
$$\cdot\left\{1+\frac{1}{2}\operatorname{cotg}^2\frac{\theta}{2}+2\lambda+\frac{1}{2}\lambda^2\left[1+(1-\beta^2)^{-1}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)\right]\right\},$$

where $\beta = v/c$ of the particle; M is the nucleon mass. $\boldsymbol{p}-\boldsymbol{p}_0$ is the change of momentum of the scattered particle; and $F(|\boldsymbol{p}-\boldsymbol{p}_0|)$ and $G(|\boldsymbol{p}-\boldsymbol{p}_0|)$ are given by

$$F(|\boldsymbol{p}-\boldsymbol{p}_0|) = (K^4/16\pi^2)\,|\mathcal{F}\{V_0(\boldsymbol{r})\}|^2,$$
$$G(|\boldsymbol{p}-\boldsymbol{p}_0|) = (K^4/16\pi^2)\,|\mathcal{F}\{A_0(\boldsymbol{r})\}|^2.$$

The limiting cases considered by PAULI and by JAUCH [5] may easily be derived from the above formulae. Introducing $\varrho(\boldsymbol{r}) = e\varrho_0(\boldsymbol{r})$, the charge density, and using (1), these two expressions may be written

$$F(|\boldsymbol{p}-\boldsymbol{p}_0|)) = |\mathcal{F}\{\varrho_0(\boldsymbol{r})\}|^2, \qquad G(|\boldsymbol{p}-\boldsymbol{p}_0|) = \frac{9\pi}{2}\left|\int \mathrm{d}\boldsymbol{r} r\varkappa(r)(Kr)^{-\frac{3}{2}}J_{\frac{3}{2}}(Kr)\right|^2.$$

The ratio G/F is unity at small angles, at large angles however it depends very strongly on the assumed charge and current distribution. The simplest assumption of a constant charge density and a linearly increasing $\varkappa(r)$ (solid nucleus) leads to (R_0 is the nuclear radius).

$$(4)\qquad F(|\boldsymbol{p}-\boldsymbol{p}_0|) = 9(KR_0)^{-3}\frac{\pi}{2}\left[J_{\frac{3}{2}}(KR_0)\right]^2,$$

$$(5)\qquad G(|\boldsymbol{p}-\boldsymbol{p}_0|) = 225(KR_0)^{-7}\frac{\pi}{2}\left[3J_{\frac{3}{2}}(KR_0)-(KR_0)J_{\frac{1}{2}}(KR_0)\right]^2.$$

In Fig. 1 we give some graphs of f_{electric} and f_{magnetic} relative to incident μ-mesons of 1 GeV for the two values $\lambda = \frac{1}{2}$ and $\lambda = 1$. As (2) shows f_{electric} is quite independent of the nucleus considered, but this is not true for f_{magnetic}, which contains the ratio G/F. However for the angles concerned in Fig. 1 the ratio G/F may contribute at the most a factor of the order of 2. The graphs in Fig. 1 have been evaluated using (4) and (5). We have chosen Li^7 as an example to be sure of the validity of the Born approximation.

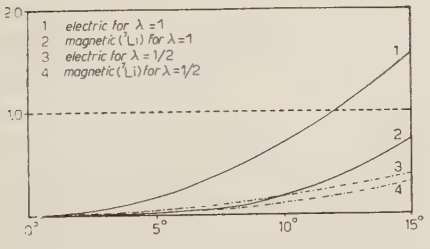

Fig. 1.

(5) W. PAULI: *Rev. of Mod. Phys.*, **13**, 203 (1941); J. M. JAUCH: *Helv. Phys. Acta*, **14**, 451 (1940).

We would like, however, to make the following remarks. It is a consistent viewpoint in field theory to assume that the anomalous magnetic moments are due to the interaction of the particle with the virtual quanta of the surrounding fields. Thus, the anomalous magnetic moment of the electron essentially results from its interaction with the virtual photons. In the same way the virtual meson cloud which surrounds a nucleon is regarded to be responsible for the anomalous magnetic moment of the nucleon. A similar situation should occur in the case of the muon. The quantitative difficulty could arise of justifying on a field theoretical basis a large value of the anomalous moment. Apart from this, the whole picture would result somewhat different from that obtained by merely adding an anomalous moment term to the Dirac wave equation for the muon. As long as small momentum transfers in the rest system of the incident meson are involved (« small », should mean, compared to some unknown inverse wavelength perhaps of the order of the nuclear dimensions), the two treatments may give fairly coincident answers, but this may no more be the case at higher energies and at larger angles where large momentum transfers are involved. In this latter case the effect of the finite dimensions of the virtual cloud responsible for the anomalous moment should be properly considered. On qualitative grounds one would expect a smaller effect than predicted by the anomalous term added to the Dirac hamiltonian. A possible theoretical explanation of the recently observed anomalous large angle scattering on the assumption of a « rigid » anomalous moment for the muon would be faced by such a difficulty.

Thanks are due to Professor B. Ferretti and to Professors G. Morpurgo and B. F. Touschek for discussions on the subject.

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RIASSUNTO

In relazione alle recenti esperienze di diffusione di mesoni μ contro nuclei, vengono valutati gli effetti dovuti al momento magnetico anomalo del mesone incidente ed alla distribuzione di correnti nel nucleo. I risultati si riferiscono a fasci non polarizzati incidenti su nuclei non polarizzati. Ad angoli relativamente grandi le sezioni di urto dipendono sensibilmente dal modo come sono distribuite le cariche e le correnti dentro il nucleo.

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Measurements of the « Noise » of Different Types of Microscope Stages.

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Summary. — The « stage-noise » of seven different microscopes (Cooke M 4000, Leitz Ortholux, Leitz Koordinatenkomparator, Zeiss Standard, Koristka M.S.2) was measured interferometrically. The variation of « stage-noise » with cell-size has been investigated, and the possibilities for the elimination of « noise » from the results of scattering measurements are considered. The minimum cell-sizes required with the different types of microscopes to obtain scattering angles significantly greater than « noise » are tabulated for protons of three energies.

1. – Introduction.

Recently it has been becoming of increasing interest to improve on, and further develop the methods for the measurement of high energies on tracks in nuclear emulsions. We mention, among others, the problems concerning meson production in nuclear encounters and the energy spectrum of the primary component of the cosmic radiation. The determination of energies higher than that corresponding to minimum ionization usually consists in the measurement of the multiple scattering since particles of these energies will ordinarily not come to rest inside the stack of emulsions (range measurements thus being not feasible), and only in very special cases will they cause secondary reactions from the analysis of which the primary energy can be deduced.

In the evaluation of scattering measurements it is of paramount import-

ance to determine the extent to which the result is affected by « spurious scattering » (« noise »). There are three groups of possible sources of « spurious scattering » (GOLDSCMIDT-CLERMONT 1950, GOTTSTEIN 1953): Those having their origin in the properties of the emulsion, those due to the observer himself, and those depending on the mechanical quality of the microscope stage and the general solidity of the instrument (summarily called « stage noise »). Whereas the contributions of the first two groups to the « spurious scattering » may be considered independent of the size of the cells in which the coordinate readings on the track have been taken this is not true, for most types of microscope stages, for the contributions of the third group. (It is also not true for the influence of distortion of the emulsion, an effect which is, however, not within the scope of the present investigation). The « stage noise » usually increases with cell-size — a possible explanation for this has been mentioned in GOTTSTEIN (1953) — and often represents, for the large cell-sizes required for the measurement of high energies, the predominant source of total « noise ». If the law for the variation of « noise » with cell-size is known it becomes possible under favourable conditions to eliminate the « noise » from the total angle of scattering measured. It has therefore been considered worthwhile to investigate the behaviour with regard to « stage noise » of the different types of microscopes which were, are, and might be used for scattering measurements by members of our laboratory.

2. – Experimental procedure.

An interferometric device was used for the control of the microscope stages which was specially designed for this purpose by Messrs. ZEISS-WINKEL, Göttingen (*). It consists of a glass-plate which has to be attached to the stage in such a manner that its optically flat, semi-silvered surface is vertical to the plane of the stage and as far as possible parallel to the direction of the stage movement, and of a 10× objective with a second, smaller glass-plate fixed in front of its front lense. The surface of this smaller glass-plate is also optically flat and semi-silvered. By means of a special connection piece this objective may be screwed, in a horizontal position, into the revolving head of the microscope so that the two flat glass-surfaces are opposite and nearly parallel to each other. The interference fringes produced between the surfaces by a Na-lamp are then observed through the ordinary binocular or monocular

(*) We are very grateful to Dr. K. MICHEL of Messrs. ZEISS-WINKEL for his kind and resourceful cooperation.

tube of the microscope, the light being bent from the horizontal direction of the objective into the vertical of the microscope tube by a small prism built into the connection piece. From the movement of the fringes taking place when the two glass-planes are displaced relatively to each other (i.e., when the stage with the attached glass-plate is moved along the plate fixed to the objective) conclusions may be drawn as to the rectilinearity of the stage transport.

The following relations hold:

$$\alpha = \frac{\lambda}{2\Delta x}, \tag{1}$$

$$\Delta d = N\frac{\lambda}{2}; \tag{2}$$

α (assumed to be small): angle between the semi-silvered glass-planes.

λ = 0.589 microns: wave-lenght of Na-light.

Δx = distance between adjacent fringes in microns.

N = number (or fraction) of fringes moving across hair-line in the eye-piece when the lateral distance between the two glass-planes is changed by Δd (in microns).

N was determined with the help of one of those eye-piece scales usually employed for reading track coordinates in scattering measurements (see fig. 1 of MENON *et al.*, 1951).

It is thus possible, by measuring both the changes in N and in Δx, to determine simultaneously the variations of Δd and of α, i.e. the « coordinate (sagitta) noise » and the « angular noise » corresponding to the « coordinate (sagitta) » (FOWLER, 1950) and the « angular » (GOLDSCHMIDT-CLERMONT *et al.*, 1948) method of measuring multiple scattering. Since the coordinate method has been predominantly employed in recent work we have limited ourselves to a systematic investigation of « coordinate noise » and have only made occasional checks on the variations of Δx.

In all, we have investigated the stage-noise of seven different microscopes: Three Cooke M 4000, one Leitz Ortholux, one Leitz Koordinatenkomparator (Kernspurmeßmikroskop), one Zeiss Standard and one Koristka M.S. 2. For the measurements on the Koristka M.S. 2 we employed an interferometer independently developed by Fratelli KORISTKA which works on essentially the same principle as that of Messrs. ZEISS-WINKEL and which is now going to be regularly supplied together with the M.S. 2 microscope.

The stages were moved from one end of the transport screws to the other and readings of the position of one arbitrarily selected fringe were made every

50 or 100 microns. The values of N and ΔN for these and larger « cell-sizes » were then calculated from these readings.

3. – Results.

Figs. 1 to 7 show the relative lateral distance (in units of 0.5 microns, in a scale with arbitrary zero point) between the semi-silvered glass-planes, as a function of the position on the stage transport screw, i.e. of the reading on the screw micrometer or the vernier, for the seven different microscopes. In the ideal case of complete absence of stage noise the curves should be straight lines. Their inclination depends on the deviation from parallelism between the direction of stage transport and the larger glass-plane and has nothing to do with the properties of the stage.

That the results obtained by the method are well reproducible is demonstrated by fig. 1. Here the stage characteristics of one Cooke M 4 000 microscope (No. 4 019) are plotted. The measurements denoted by curves A and B were made on two different days, by two different observers progressing in opposite directions on the screw (indicated by arrows) and with changed alignment between glass-plate and screw direction. The two curves show essentially the same features.

The stage movement of two further Cooke microscopes is plotted in figs. 2 and 3: of M 40 135, said to be the best « scattering microscope » at Bristol University, and of M 4 074 which this investigation revealed to be in a rather poor state of repair. Consequently the latter was overhauled and is now in a much better condition (Menon, private communication).

Fig. 4 gives the corresponding information for a Leitz Ortholux microscope which had been in use for several months in our laboratory. While

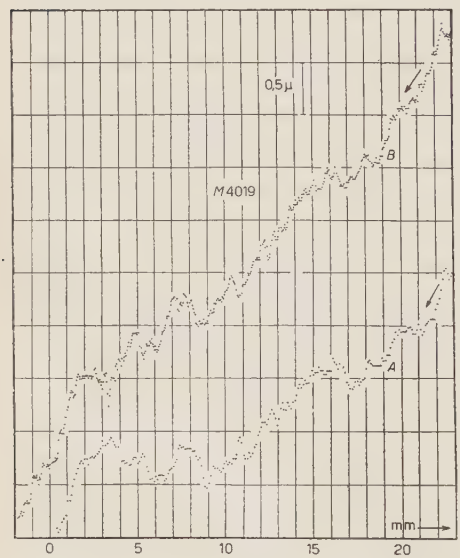

Fig. 1. – Stage movement of microscope Cooke M 4019. For significance of A and B see text. Between 0 and 18 mm, curve A should be displaced to the left by about 400 μ (systematic error in micrometer reading). Occasional discrepancies between the two curves (e.g. at 3.5 mm) may be explained as « jumping a fringe ».

the readings represented by curve *A* were taken the instrument was slightly knocked. This happened when the stage was at position 119 mm. In order to check whether the change in the inclination of curve *A* occurring at this point was merely due to this disturbance the measurements were repeated later on and resulted in a curve *B* roughly parallel to the latter part of *A*.

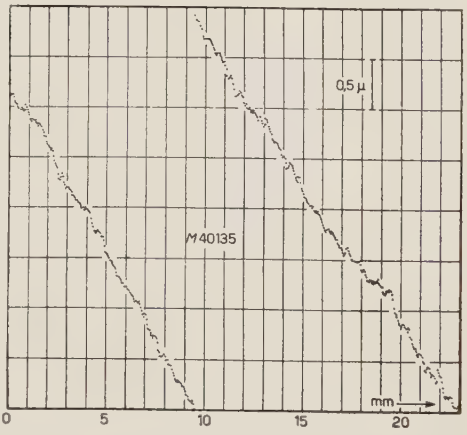

Fig. 2. – Stage movement of Cooke M 40135.

The stage of the Leitz Koordinatenkomparator was also controlled on the entire length of its traverse (150 mm) but to save space we only reproduce the curve for a typical section of 38 mm length (fig. 5).

The corresponding data for one Zeiss Standard microscope are given in fig. 6. Whereas the stage transport is very far from rectilinear for the first 20 mm of the motion its rectilinearity is comparable to that of the other microscopes for the last 20 mm.

The stage movement of the Koristka M.S. 2 is plotted in fig. 7. At first, the screw positions 0 and 15 mm, representing a considerable stage-noise (curve *A*). Consequently one cog-wheel was replaced which resulted in a remarkable improvement (curve *B*) (*).

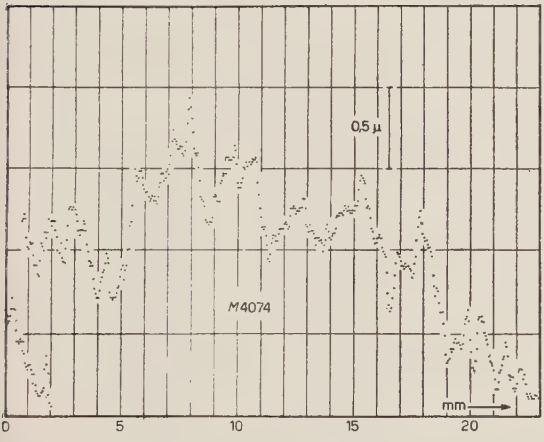

Fig. 3. – Stage movement of Cooke M 4074 before overhauling.

While figs. 1-7 convey a general impression of the quality of different sharp peaks to the curve with a period of about 0.5 mm were observed between stages it is necessary to arrive at a quantitative estimate of the contribution of « noise » to « signal » and of the

(*) The results for the Ortholux and the 16 to 27 mm-part of the Koristka M.S.2 are in rather good agreement with those obtained by J. K. Bøggild and M. Scharff on microscopes of these two types by the « Reversing Sagitta Method » (Scattering

variation of this contribution with « cellsize ». For this purpose the mean second differences D due to stage-noise have been calculated from the interferometric readings for different « cellsizes » c. The results are shown in figs. 8 and 9. While the stage-noise is smallest for the Koristka M.S. 2 and (in that order) the Leitz Ortholux it appears that the variation of stage-noise with cell-size is roughly the same for all types of microscopes, with the exception of the Koristka M.S. 2 the stage-noise of which remains approximately constant for cell-sizes between 200 and 2 000 microns. For the Leitz Koordinatenkomparator it is of significance whether the basic cell-size (the size of the steps by which the stage is actually moved on) is smaller than 100 microns, or 100 microns and larger. In the former case the necessity of touching

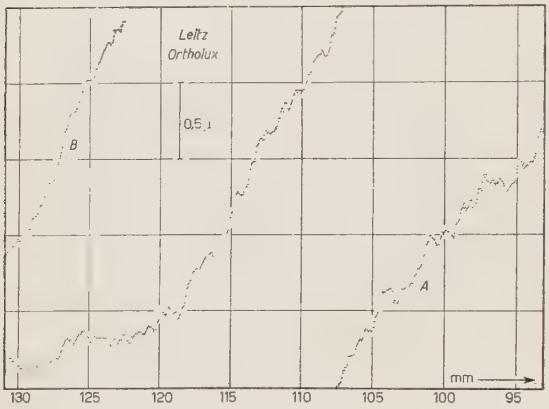

Fig. 4. – Stage movement of Leitz-Ortholux No. 448573. The instrument was slightly knocked when position « 119 mm » had been reached (curve A). Repeated measurements yielded curve B.

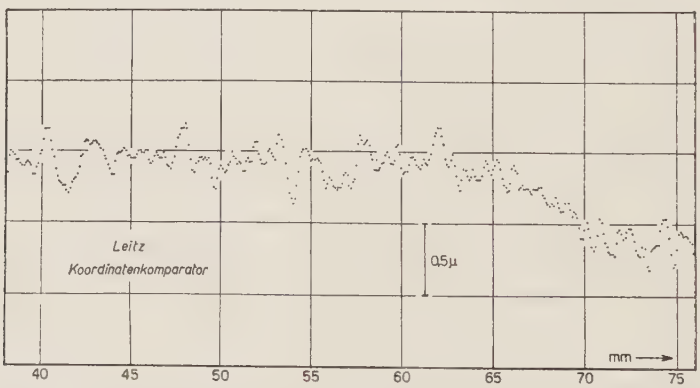

Fig. 5. – Stage movement of the Leitz Koordinatenkomparator (y-axis) for a typical section of the stage trasverse.

measurement by looking at the track both from the emulsion side and through the glass backing. Adding of the coordinates so obtained yields the « stage-noise curve », subtraction from each other the shape of the track). There is, however, an indication that the stage-noise is slightly smaller in our cases. We thank Dr. Bøggild and Mr. Scharff for the communication of their results prior to publication.

the micrometer screw of the reading microscope results in a higher stage-noise (*).

FOWLER (private communication) measuring the scattering of tracks of very energetic particles on Cooke microscope M 4019 has found a variation of total noise (D_{noise}) with cell-size c as $c^{0.25}$. Moreover, by subtracting the contribution of « true » scattering (as derived from the measurements in very large cells) from the scattering measured in short cells (where the contribution of stage-noise to the total noise is still unimportant) he obtained a value of about 0.11 microns for the cell-size-independent part of the total noise. In order to check the consistency between these results and the values of the stage noise as obtained by the interferometric method we added a constant value of 0.11 microns to the latter. Whereas $D_{\text{stage noise}}$ alone in this case varies roughly as $c^{0.6}$ the resulting values of $(D^2_{\text{stage noise}} + 0.11^2)^{\frac{1}{2}}$ are well represented by a $c^{0.25}$-law (fig. 10).

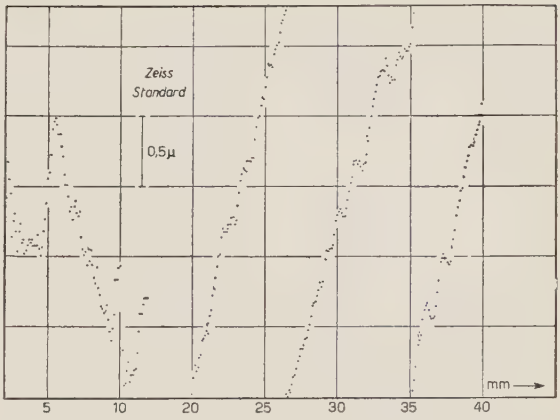

Fig. 6. Stage movement of Zeiss-Standard No. 291796.

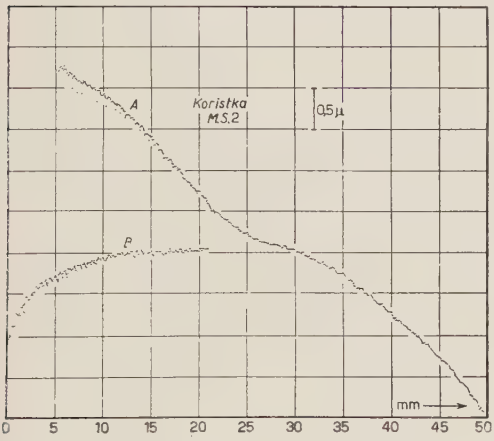

Fig. 7. – Stage movement of Koristka M.S.2. Before (A) and after (B) replacement of cog-wheel.

As mentioned in **1**, there is a possibility of eliminating noise by measuring the scattering for at least two different cell-sizes c_1 and c_2, provided the law for its variation with cell-size is known. I.e., it

(*) *Note added in proof*: F. D. HÄNNI (private communication) has recently carried out stage-noise measurements on another Leitz Koordinatenkomparator which is in use at Berne University. By means of our interferometric device he controlled both the x- and y-movement of the stage and obtained results in good agreement with ours.

is required to know the quantity m in the following equations:

$$(3) \qquad \overline{D}^2_{\underset{c}{\text{Total}}} = \overline{D}^2_{\underset{100}{\text{True}}} s^3 + u s^m$$

$$(4) \qquad \overline{D}^2_{\underset{100}{\text{True}}} = \frac{\overline{D}_1^2 - \overline{D}_2^2 (s_1/s_2)^m}{s_1^3 - s_2^3 (s_1/s_2)^m} .$$

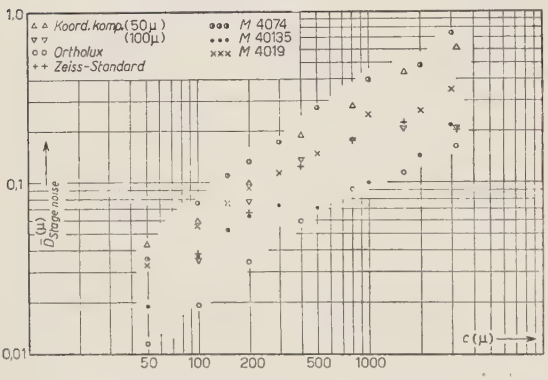

Fig. 8. – The « stage-noise scattering » (mean second difference $\overline{D}_{\text{stage noise}}$) as a function of cell-size c for six different microscopes.

(The dependence of the logarithmic term in the « scattering constant » on cell-size (GOTTSTEIN *et al.*, 1951) has been neglected in the derivation of these formulae). $D_{\underset{c,\,1,\,2,\,100}{\text{True, Total}}}$ is the true or total, respectively, mean second difference of scattering for the cell-sizes c, c_1, c_2 or 100 μ (a convenient standard cell-size), respectively

$$s = c/100 \text{ microns.}$$

$$u = \text{const.}$$

Table I contains the values of m for the different microscopes, for a cell-size-independent contribution to total noise of 0.1 microns.

TABLE I.

Microscope	Cell-size-region	m
Cooke M 40135	50 – 1 000	~ 0.25
	1 000 – 3 000	~ 1.0
Cooke M 4019	50 – 3 000	~ 0.5
Koordinatenkomparator . .	100 – 800	~ 0.6
	800 – 3 200	~ 0.25
Ortholux 448573	50 – 400	~ 0.125
	400 – 3 200	~ 0.5
Zeiss Standard	50 – 3 200	~ 0.5
Koristka M.S.2.	50 – 1 600	~ 0.0
	1 600 – 3 200	~ 1.0

The division into cell-size regions within which m is considered to be approximately constant is of course rather arbitrary. In reality m will change slowly but continously with cell-size. It is, however, sufficient to know the order of magnitude of m since considerable alterations in m affect only slightly the value of $D_{\underset{100}{\text{True}}}$ as calculated from (4).

The occasional checks mentioned in 2, on the average distance between adjacent fringes showed that the maximum changes in angle between the two semi-silvered glass surfaces which occurred when the stage was moved along the whole of its traverse were of the order of 0.01 degrees for all the Leitz, Zeiss and Koristka microscopes and for Cooke M 40 135. It must be kept in mind that these angular changes may be due to rotations of the stage round any of the three spatial axes whereas for the « angular stage noise » which is influencing the results of angular scattering measurements it is only the rotations round an axis parallel to the optical axis which matter. The angular deviations measured by the interferometric method represent, therefore, an upper limit to the angular stage noise.

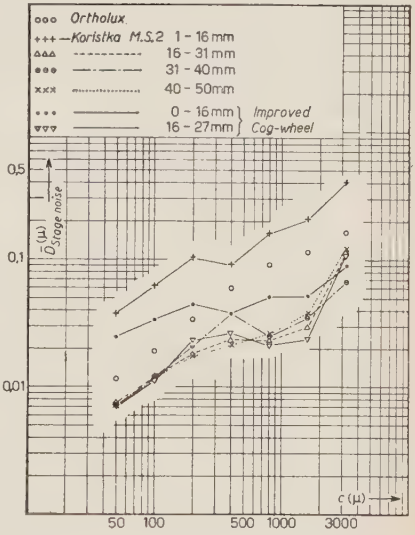

Fig. 9. – The « stage-noise scattering » (mean second difference $\overline{D}_{\text{stage noise}}$) as a function of cell-size c for different parts of the screw of the Koristka M.S.2. In order to facilitate comparison with the results shown in Fig. 8 the points for the Leitz Ortholux have been reincluded.

4. – Discussion and Conclusions.

According to the results given in 3. The different microscopes investigated here show somewhat different characteristics with respect to (« coordinate ») stage-noise. In principle, however, any microscope with a movable stage — whatever its stage noise — may be used successfully for scattering measurements on tracks of arbitrary straightness provided that the track length available allows one to employ cells large enough for the « true » scattering to become greater by a certain factor (four, e.g.) than the total noise, and to have enough of these cells for the statistical error to remain within bearable limits. Since the track length available is limited by the geometry of the

block of emulsions the usefulness of a microscope for scattering measurements is given by how far it allows one to reduce the cell-size on a track of a particle of given energy without yielding a value for the scattering smaller than, say, four times the total noise. Fig. 11 shows the values of $4(D^2_{\text{Stage noise}} + 0.1^2)^{\frac{1}{2}}$ as a function of cell-size c for the different types of microscopes (only the best of the three Cooke microscopes having been included). The straight lines denote $\bar{D}_{\text{True}}$ for protons of 100, 1 000 and 10 000 MeV energy. Their intersections with the curves for $4(D^2_{\text{Stage noise}} + 0.1^2)^{\frac{1}{2}}$ give the minimum cell-sizes required to obtain a « true » scattering equal to, or greater than, four times the total noise. The results are collected in table II.

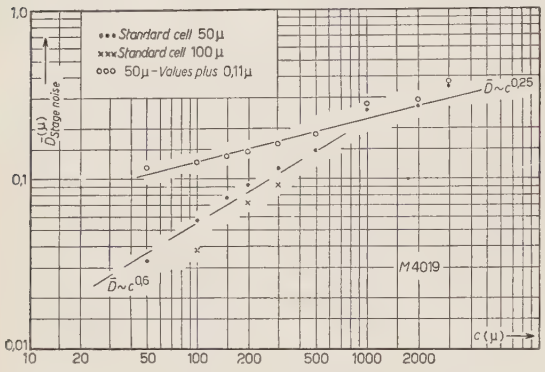

Fig. 10. – $\bar{D}^2_{\text{stage noise}}$ and $(\bar{D}_{\text{stage noise}} + 0.11^2)^{\frac{1}{2}}$ as a function of cell-size for Cooke M 4019.

TABLE II. – *Minimum cell-size (microns) required to obtain « true » scattering equal to, or greater than, 4 times the total noise.*

Proton energy	100 MeV	1 000 MeV	10 000 MeV
Cooke M 40135	160	680	ca. 3 500
Koordinatenkomparator	165	900	ca. 3 500
Ortholux 448573	150	660	3 100
Zeiss-Standard	160	900	ca. 3 500
Koristka M.S.2.	150	580	2 400

It follows that the differences between the required minimum cell-sizes for the different microscopes are not too great though noticeable. Measuring, e.g., the scattering of a 1 000 MeV-proton of given track length on a Zeiss-Standard and a Koristka M.S. 2 one would have at one's disposal in the latter case about 1.5 times as many cells than in the former one and the result would, correspondingly, be subject to a statistical error smaller by a factor of about 1.25.

The differences in accuracy attainable with the different types of instruments being so relatively small it appears that in choosing a certain model for permanent use in nuclear emulsion work it would be wise to consider not only

its stage noise but also the facilities it offers for speedy and convenient work, i.e., the length of stage traverse available, the possibilities to rotate the plate and to read accurately the stage coordinates, etc., not to speak of the optical requirements and the price.

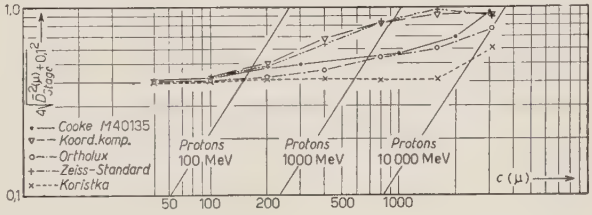

Fig. 11. - Four times the total noise as a function of cell-size c. The straight lines denote $\overline{D}^2_{\text{True}}$ for protons of different energies.

We are much indebted to Mr. P. H. Fowler for valuable discussions and his assistance with the measurements on the Cooke M 4019 microscope, to Dr. Cantù for his help with the work on the Koristka M.S. 2, to the firms of Leitz, Wetzlar, and Zeiss-Winkel, Göttingen, who placed, respectively, at the disposal of our institute a Koordinatenkomparator and a Zeiss Standard microscope, and to Fratelli Koristka for the hospitality extended to the author at their works in Milan.

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RIASSUNTO (*)

Lo « stage-noise » di sette differenti microscopi (Cooke M 4000, Leitz Ortholux-Leitz Koordinatenkomparator, Zeiss Standard, Koristka M.S.2) è stato misurato interferometricamente. Si studiano la variazione dello « stage-noise » colla dimensione della cella e si considerano le possibilità per la eliminazione del disturbo dai risultati delle misure di scattering. Le minime dimensioni delle celle richieste dai differenti microscopi per ottenere angoli di scattering significantemente maggiori del disturbo sono tabulate per protoni di tre energie.

(*) *Traduzione a cura della Redazione.*

On Cosmic Rays Jets.

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(ricevuto il 3 Agosto 1954)

Summary. — 29 jets found in the plates which have been launched at Cagliari in summers of 1952 and 1953 have been studied. The mean energy for mesons emitted in each jet is calculated after introduction of the anelasticity coefficient; it is inferred that the values for said energy in the low multiplicity jets are slightly greater than in high multiplicity jets. Angular distribution in the jets is studied; the results do not disagree with an isotropic distribution.

Introduction.

Results are referred in this work of the analysis of a group of jets found in Ilford plates G5 exposed to cosmic radiation for some hours, at the height of about 22 km at Cagliari, during flights effected in the summers of 1952 and 1953. Some partial results have been previously published in two short communications ([1,2]).

Different authors have recently developed particularized methods ([3-6]) for

([1]) G. Bertolino, A. Debenedetti, G. Lovera and M. Vigone: *Nuovo Cimento*, **10**, 991 (1953).

([2]) G. Bertolino, D. Pescetti and L. Tallone: *Padua Meeting*, April 1954.

([3]) C. C. Dilworth, S. J. Goldsack, T. F. Hoang and L. Scarsi: *Compt. Rend.*, **236**, 1551 (1953).

([4]) T. F. Hoang: *Journ. de Phys.*, **14**, 395 (1953).

([5]) C. C. Dilworth, S. J. Goldsack, T. F. Hoang and L. Scarsi: *Nuovo Cimento* **10**, 1261 (1953).

([6]) C. Castagnoli, G. Cortini, C. Franzinetti, A. Manfredini and D. Moreno: *Nuovo Cimento*, **10**, 1539 (1953).

calculating the energy of the primary in the center of mass frame of the two colliding nucleons (said frame will be hereinafter shortly indicated as C.M.), starting substantially from the hypothesis that the mesons of the jet are produced in a single nucleon-nucleon collision. The calculation of energy is obtained from angular measurements on the jet particles. The results which are obtained by using the different methods are very near each other and, generally, are coincident in the limits of statistical errors. The value of the primary energy in the C.M. frame calculated in this manner is at the basis of the most recent works on jets ([3-7]) and of this work.

It was not possible to verify the « tunnel » theory ([8]) because we have not sufficent experimental material among the jets studied in this work.

1. – Dependance of Multiplicity from the Primary Energy in the C.M. Frame.

The jets studied in the present work are 29, 20 of which from a single charged primary, 7 from a neutral primary and 2 from an α-particle. The jets studied have at least 5 shower tracks and no more than 4 grey or black tracks.

The calculation of the primary energy in the C.M. frame, has been effected using the formula

$$\frac{1}{\gamma_c^2} = \operatorname{tg} \vartheta_f \cdot \operatorname{tg} \vartheta_{1-f} , \tag{1}$$

wherein γ_c is the primary energy in unity of rest energy Mc^2; ϑ_f and ϑ_{1-f} are the half widths of the cones which, in the laboratory frame comprehend fraction f and $1 - f$, respectively, of the jet particles.

The formula (1) has been calculated on the following hypothesis:

1) The jet is assumed to be symmetrical in the C.M. frame with respect to the line of flight of the incident particle and to the plane perpendicular to this line and passing at the center of mass. This assumption is true in the nucleon-nucleon collision.

2) The velocities $c\beta_\pi$ of mesons produced in the C.M. frame are of the order of the velocity $c\beta_c$ of the center of mass in the laboratory frame, which hypothesis is satisfied if both the center of mass velocity and the mesons velocities broadly approach the velocity of light.

Fig. 1 plots the most probable number

$$n_\pm = n_s - 1$$

([7]) T. F. Hoang: *Journ. de Phys.*, **15**, 22 (1954).

([8]) F. C. Roesler and C. B. A. McCusker: *Nuovo Cimento*, **10**, 127 (1953).

of mesons produced as a function of γ_c. The jets from a single-charged primary are marked with a circle, while the jets from a neutral primary are marked with a dot, and the jets from an α-particle are marked with a star. The study of the data plotted in Fig. 1 leads to the broad subdivision of the jets in two groups, which are indicated in the plots as « zone A » and « zone B ». The jets of said two zones correspond to the jets indicated by the above mentioned authors as « high multiplicity » and « low multiplicity » jets respectively. In a recent work ([3]) DILWORTH *et al.* have suggested an interesting hypothesis for explaining the subdivision of the jets in two groups as due to a difference between the particles producing the jet. (See also DEMEUR, DILWORTH, SCHÖNBERG, *Nuovo Cimento* (1952), where said hypothesis was suggested). We do not discuss in this work this hypothesis.

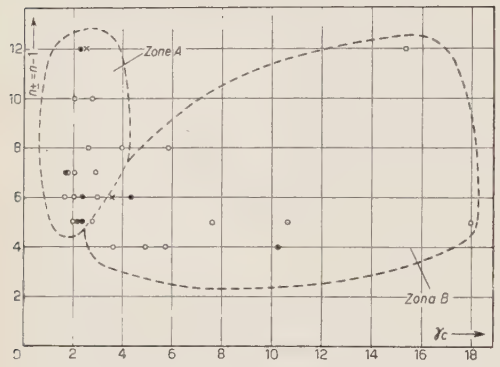

Fig. 1. – ○ p-jets; ● n-jets; × α-jets.

No difference is noted in this plot between the jets with neutral primary and the jets with single-charged primary.

2. – Anelasticity Coefficient.

The anelasticity coefficient K, defined as the ratio between the total energy of the emitted mesons and the kinetic energy of nucleons before collision in the C.M. frame, is given with a reasonable approximation by the following formula:

$$K = \frac{1.5 n_{\pm} \cdot \gamma_c \cdot \sin \vartheta_L}{2\mu(\gamma_c - 1)}, \tag{2}$$

which has been calculated starting from the hypothesis:

1) all the mesons produced in the jet are π-mesons;

2) the number of neutral π-mesons emitted equals one half of the number of charged π-mesons;

3) the mesons in the jet are emitted with energies which are not substantially different from each other and with $\beta_\pi \cong \beta_c$.

In formula (2) μ indicates the ratio between the mass of the nucleon and the mass of the π-meson, ϑ_L is the limiting angle of the jets in the laboratory frame, $n_\pm$ is the most probable number of charged mesons in the jet, γ_c the primary energy.

The value of K has been calculated both for jets with single-charged primary and with neutral primary, and variation of K has been studied as a function of the energy of the primary in the C.M. frame.

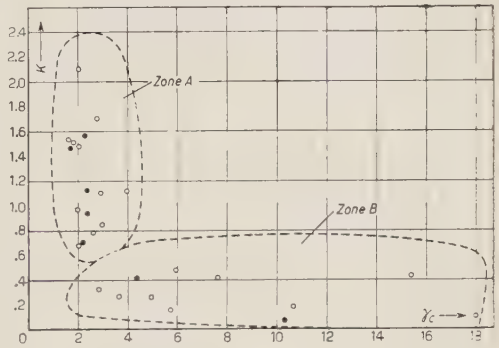

Fig. 2. ○ p-jets: ● n-jets.

The results obtained are plotted in Fig. 2, in which the anelasticity coefficient is a function of γ_c. The subdivision of the plotted points in two zones is clearer than in Fig. 1. Jets of small value of γ_c are plotted in zone A, corresponding to zone A of Fig. 1, and are characterized by an high value of the anelasticity coefficient, while jets of high value of γ_c are plotted in zone B, corresponding to zone B of Fig. 1, and are characterized by a small value of the anelasticity coefficient. It is remarked that no jet of high energy ($\gamma_c > 4$), among the jets studied in this work, falls in zone B.

The mean value of K for the jets of the two zones are respectively

$$\text{Zone } A \qquad \overline{K}_A = 1.17 \pm 0.33 ,$$

$$\text{Zone } B \qquad \overline{K}_B = 0.28 \pm 0.13 .$$

The fact that the value calculated for the anelasticity coefficent for the jets of zone A is greater than 1, while for its meaning the anelasticity coefficent must be $K \leqslant 1$, leads to the conclusion that the scheme introduced for explaining the jet production is quite incomplete. We think that it can be reasonably assumed that for the jets of zone A

$$K_A \simeq 1$$

and we infer that the value of K_A greater than 1 calculated, which value is however in the limits of statistical fluctuations, can be interpreted simply as a fluctuation or as a consequence of secondary interactions between the particles emitted in the nucleon-nucleon collision and the nucleon of the target nucleus.

The ratio $C = n_\pm / \gamma^{1/4}$ has been studied, as previously by HOANG for jets

characterized by a small anelasticity coefficent (γ' indicates the energy of the primary in the laboratory frame). The value of C calculated on the base of our few data broadly approaches to a mean value

$$\bar{C} = 1.81 \pm 0.62 ,$$

wherein 0.62 is the root mean square of the deviations. As a consequence the empirical law providing the number of charged mesons as a function of the energy is

$$n_{\pm} = (1.81 \mp 0.62)\gamma'^{\frac{1}{4}} .$$

The numerical coefficent approaches the coefficent calculated by HOANG (1.87 ± 0.35).

The most probable number $n_{\pm}$ of mesons emitted is plotted in Fig. 3 as a function of $K(\gamma_c - 1)$, that is as a function, less a factor 2, of the part of the energy ΔE which, in the C.M. frame, is absorbed in meson production. The study of this plot shows that the division of the mesons in two zones as observed in the plots of Fig. 1 and 2 can no longer be observed. In Fig. 3 the jets of zone A are indicated by a circle and the jets of zone B by a cross. The analysis of the results obtained shows that the anelasticity coefficent acts so that the mean energy for meson in the C.M. frame is in the low multiplicity (and low anelasticity) jets not considerably greater than in high multiplicity (and high anelasticity) jets. More particularly the mean value of the energy for each meson emitted in the C.M. frame is for high multiplicity jets (zone A) 301 ± 100 MeV and the mean value for low multiplicity jets (zone B) is 383 ± 180 MeV, wherein 100 and 180 respectively are the root mean squares of the deviations. Said results substantially agree with the values calculated from HOANG's data (402 and 482 MeV respectively). It is noted that the ratios between the mean values of the energy of the mesons emitted in the low and high multiplicity jets respectively calculated from the data deduced from HOANG's work and from our data are nearly equal, that is 1.21 and 1.27 respectively.

By calculating the mean values of the energies plotted in Fig. 3 for each horizontal line, which corresponds to calculate the mean value of the total energy for jets having the same number of tracks, we draw the diagram of Fig. 4. The dots of said diagram are arranged in a satisfactory manner along a straight line (indicated by dots in the diagram), the equation of which, calculated by the method of minimum squares, is

$$n_{\pm} = 2.45\,[K(\gamma_c - 1)] + 2.53 . \tag{3}$$

Said results seems to agree both with the theory of HEISENBERG (9) and with the theory of WATAGHIN (10-12).

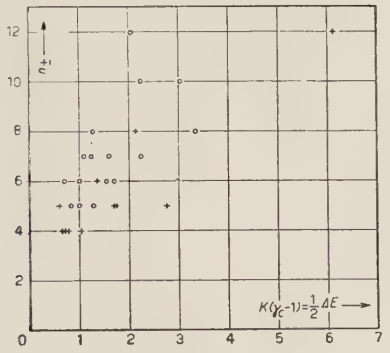

Fig. 3. – ○ jets of Zone A; + jets of Zone B.

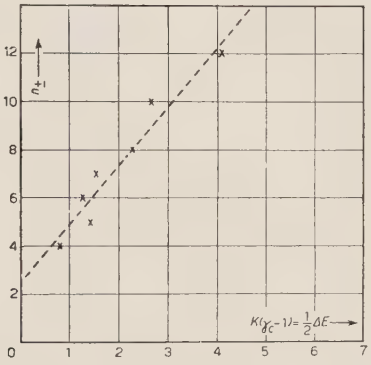

Fig. 4.

3. – Angular Distribution of Produced Mesons in the C.M. Frame.

The study of the angular distribution of produced mesons in the C.M. frame has been effected subdividing, according to the method followed by CASTAGNOLI *et al.* (6), the solid angle in ten equal parts, the line of flight of the colliding nucleons being the simmetry axis. The number of tracks falling in each of said ten parts of the solid angle has been calculated.

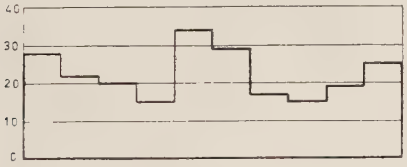

Fig. 5. – p-, n-, α-jets (227 tracks).

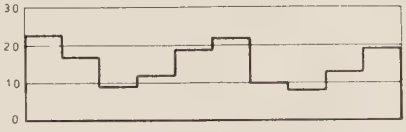

Fig. 6. – p-jets (152 tracks).

The results obtained for all the jets considered are plotted in the diagram of Fig. 5 (27 jets with a total of 224 tracks), while the diagrams of Fig. 6, 7, 8 show separately the angular distribution for p-jets, n-jets, α-jets.

The diagram of Fig. 6 (20 jets with a total of 152 tracks) altough not in contradiction with an isotropic distribution, would infer a distribution in which

(9) W. HEISENBERG: *Nature*, **146**, 65 (1949); *Zeits. f. Phys.*, **126**, 569 (1949).
(10) G. WATAGHIN: *An. Ac. Bras. de Ciencias* (Symposium 1941).
(11) G. WATAGHIN: *Phys. Rev.*, **74**, 975 (1948).
(12) G. WATAGHIN: *Phys. Rev.*, **75**, 693 (1949).

the highest number of mesons is emitted in the direction of the line of flight of the colliding nucleons and in the direction perpendicular to it. We think however that we are dealing with statistical fluctuations for, summing the diagram of Fig. 6 with the like diagram obtained by CASTAGNOLI *et al.*, we draw the diagram of Fig. 9 in which the above mentioned anomaly disappears.

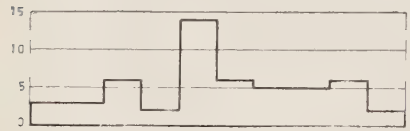

Fig. 7. – n-jets (52 tracks).

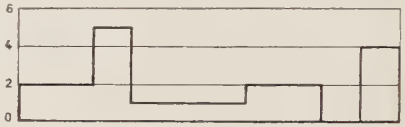

Fig. 8. – α-jets (20 tracks).

The diagram of Fig. 7, n-jets (7 jets with a total of 52 tracks) cannot be considered, in view of its low statistical weight, in contradiction with an isotropic distribution. It should however be observed that a distribution similar to the distribution plotted in Fig. 7, in which the highest number of mesons is emitted in a perpendicular to the line of flight of the colliding nucleons, has been found by CASTAGNOLI *et al.*, for n-jets. Fig. 10 sums the data of Fig. 7 and the data for n-jets found by CASTAGNOLI *et al.*, for a total number of 102 tracks.

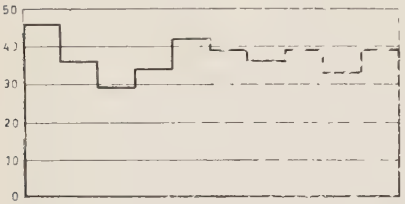

Fig. 9. – p-jets (152+221). The data of this diagram are deduced summing CASTAGNOLI *et al.*'s data (221 tracks) with our data (151 tracks).

Referring to α-jets we cannot plot a diagram of whatever a meaning from the material studied. Summing our data to the data of CASTAGNOLI *et al.* we plot the diagram of Fig. 11 for a total number of 142 tracks. The distribution can be considered isotropic also in this case in the limits of statistical fluctuations.

In a recent work DULLER and WALKER ([13]) have shown that, considering an isotropic distribution, in the case of $\beta_c = \beta_\pi$, wherein β_c and β_π are the velocity of the center of mass in the laboratory frame and the velocity of the

Fig. 10. – n-jets (52+50). The data of this diagram are deduced summing CASTAGNOLI *et al.*'s data (50 tracks) with our data (52 tracks).

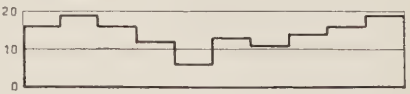

Fig. 11. – α-jets (20+122R). The data of this diagram are deduced summing CASTAGNOLI *et al.*'s (122 tracks) with our data (20 tracks).

([13]) N. M. DULLER and W. D. WALKER: *Phys. Rev.*, **93**, 215 (1954).

emitted mesons in the C.M. frame respectively, the diagram obtained plotting $\log f/(1 - f)$ as a function of $\log \mathrm{tg}\,\vartheta$ is a straight line with slope 2. In the above formula f is the fraction of the mesons contained in the angle ϑ in the laboratory frame.

The following relation is deduced from the cited work

$$\log \frac{f}{1-f} = 2 \log \mathrm{tg}\,\vartheta + 2 \log \gamma_c \tag{4}$$

from which it is inferred

$$\log \frac{f}{1-f} = 2 \log [\gamma_c \cdot \mathrm{tg}\,\vartheta]\,, \tag{5}$$

which means that the relation between $\log f/(1-f)$ and $\log [\gamma_c\ \mathrm{tg}\,\vartheta]$ is a straight line with slope 2 passing through the origin.

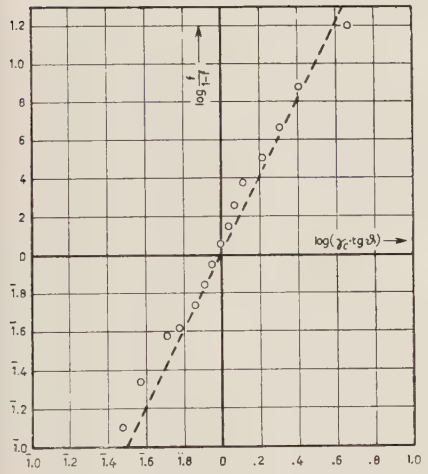

Fig. 12.

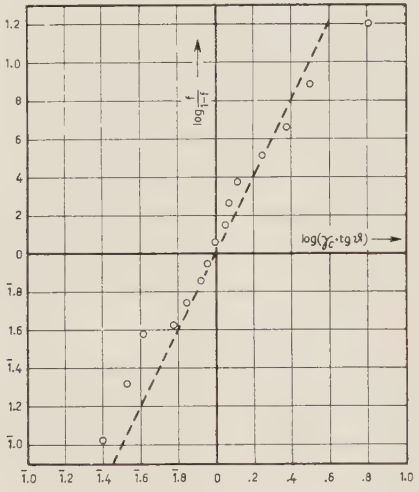

Fig. 13.

The above formula is particularly useful as it makes possible the study in one plot only, of the data concerning jets of different energies. In this manner we have drawn the diagram of Fig. 12 in which the results obtained for p, n, α-jets are summed and the plot of Fig. 13 in which the results for p-jets only are plotted. The conclusion which can be drawn from the analysis of the above plots, as could be easily foreseen, is coincident with the previous conclusions. The distribution of points in Fig. 12 and 13 indicates that the angular distribution is nearly isotropic.

We are indebted with Prof. G. WATAGHIN, who has suggested this work, and to Prof. G. LOVERA, for his constant interest and useful discussion.

We would also thank the persons who have worked for the organisation of the Sardinia expeditions of 1952 and 1953 and more particularly Prof. POWELL.

RIASSUNTO (*)

Si sono studiati 29 jets trovati nelle lastre lanciate a Cagliari nelle estati 1952 e 1953 L'energia media dei mesoni emessi in ogni jet è stata calcolata dopo l'introduzione del coefficiente di anelasticità; se ne deduce che i valori della suddetta energia nei jets a bassa molteplicità è leggermente superiore che nei jets ad alta molteplicità. Si studia la distribuzione angolare nei jets; i risultati non sono in disaccordo con una distribuzione isotropica.

(*) *Traduzione a cura della Redazione.*

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Investigations on the Reaction $^7Li(\gamma, \alpha)^3H$.

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(ricevuto il 9 Agosto 1954)

Summary. — The reaction $^7Li(\gamma, \alpha)^3H$ is studied by exposing lithium-loaded nuclear emulsions to the radiation from a 31 MeV betatron. The cross-section as a function of γ-ray energy is measured and excited states of ^7Li are found at 4.7, 5.5, 6.8, 8.3, 9 MeV. The first three levels have a relative intensity of 1 : 0.75 : 0.75. The angular distribution analysis allows one to exclude certain possible values of the spin for the levels at 4.7, 5.5 and 6.8 MeV. The absolute value of the cross-section is obtained from the known $^{12}C(\gamma, 3\alpha)$ cross-section data. A selection rule which limits the number of low energy levels is discussed.

1. – Introduction.

The aim of these investigations was to study the energies E^*, spins J and parities π of the low-lying levels of ^7Li by means of the reaction $^7Li(\gamma, \alpha)^3H$; $Q = -2.465$ MeV (Titterton, 1950 [1]). Lithium-loaded Kodak NT1a nuclear emulsions (200 μ thick) have been exposed to the γ-radiation from a 31 MeV betatron. The direction of the γ-ray relative to the plate was defined within $\pm 3°$. Neutrons, which could induce the reaction $^6Li(n, \alpha)^3H$ (Chadwick 1935 [2], Amaldi 1953 [3]) in the emulsions, have been absorbed by a 3 cm layer of boric acid. Good discrimination between the tracks of α-particles

(1) E. W. Titterton: *Proc. Phys. Soc.*, **63** A, 915 (1950).
(2) J. Chadwick and M. Goldhaber: *Nature*, **135**, 65 (1935).
(3) E. Amaldi *et al.*: *Proc. Roy. Soc.*, **149**, 522 (1935).

and tritons arising from ^{7}Li was achieved by a special development for the energy interval $2 \leqslant E_\alpha + E_{^3\mathrm{H}} < 5$ MeV.

The range-energy relation given by ROTBLAT (1) has been used for α-particles, whereas for tritons the known proton range-energy relation (4) was transformed by means of the BLACKETT formula (5). The validity of the latter has been controlled as follows: More than three hundred triton tracks of the slow neutron induced reaction ^{6}Li(n, α)^{3}H have been measured, and the mean track length R found to be

$$R = 34.8\ \mu\,.$$

For this triton range, the energy relation gives the value $E_{^3\mathrm{H}}$=2.70 MeV. As the reaction has an energy release of 4.78 MeV, the triton energy is well-defined:

$$E_{^3\mathrm{H}} = \tfrac{4}{7} \cdot 4.78\ \mathrm{MeV} = 2.73\ \mathrm{MeV}\,.$$

This value only differs by 0.03 MeV from the one found above. Thus, the range-energy relation may be used with sufficiently high precision.

The number of measured tracks was corrected for escape.

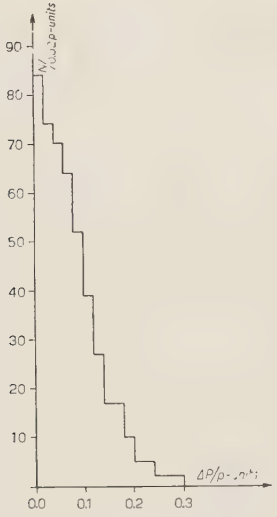

Fig. 1. – Δp-distribution. $\Delta p = |\boldsymbol{p}_\alpha - \boldsymbol{p}_{^3\mathrm{H}}|$ = momentum unbalance vector. By definition: 1 p unit = momentum of a 1 MeV α-particle; N = Track number.

Fig. 1 showing the Δp-distribution of 480 tracks enables the accuracy of our measurements to be controlled. Events for which $\Delta p > 0.20$ p-units have not been considered.

Numerous tracks, which originate from the reactions ^{14}N(γ, α)^{10}B and ^{16}O(γ, α)^{12}C, induced in the photographic emulsion (6) have also been found. Some of them could not be distinguished clearly from the ^{7}Li(γ, α)^{3}H-recoil tracks, neither by grain density difference nor by the momentum criterion. These cases are included in Fig. 2 and represented by the shaded parts.

2. – Cross-section and Energy Levels.

The three well-defined maxima in Fig. 2 indicate ^{7}Li-levels at $E^* = 4.7$, 5.5, 6.8 MeV, with the branching ratios 1 : 0.75 : 0.75.

(4) J. ROTBLAT: *Nature*, **167**, 550 (1951).
(5) M. S. LIVINGSTON and H. A. BETHE: *Rev. Mod. Phys.*, **9**, 245 (1937).
(6) C. H. MILLAR and A. G. W. CAMERON: *Canad. Journ. of Phys.*, **31**, 723 (1953).

The known 7.4 MeV level ([7]) is indicated, but cannot be resolved because of overlapping with the 6.8 MeV state.

Preliminary results (STOLL 1953 ([8])) did not show any track from the 4.6 MeV level due to insufficient development technique at this time. Also, a sharp peak occurred at 7.25 MeV due to $^6Li(n, \alpha)^3H$ contribution.

A slight rise of the cross section is observed at $E^* = 8.3$ MeV and $E^* = 9.0$ MeV.

The level observed at $E\gamma = 4.7 \pm \pm 0.1$ MeV may be identified with the one found by GOVE and HARVEY from the $^7Li(p, p')^7Li^*$-reaction ([9]) at $E\gamma = 4.67 \pm 0.08$ MeV. These authors did not observe γ-radiation from this level and therefore suggested the disintegration into α and ^3H. Their suggestion is now confirmed by the occurrence of this level in our experiment.

The 5.5 ± 0.3 MeV level we observed has not been reported previously, though search has been made for levels in the energy region between 0 and 8 MeV by different authors.

We shall therefore try to explain the absence of this level in other experiments.

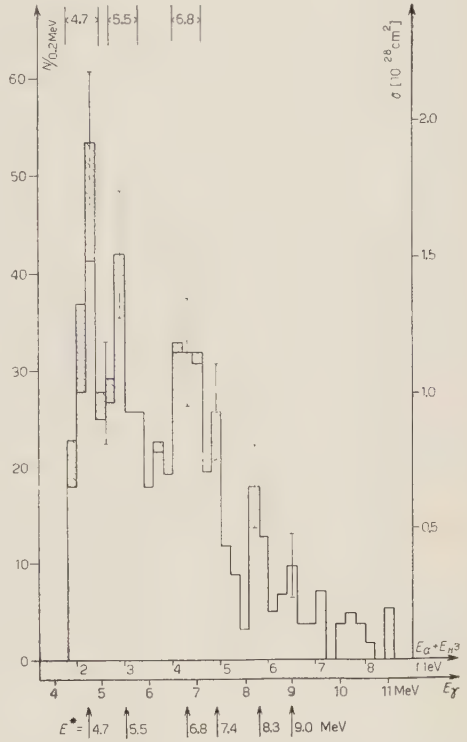

Fig. 2. – The $^7Li(\gamma, \alpha)^3H$ cross-section σ vs. γ-energy. $E^* = E_\alpha + E_{^3H} - Q$. N = Track number; I = mean statistical error.

ASHMORE and RAFFLE ([10]) investigated the reaction $^9Be(d, \alpha)^7Li$ using 5.31 MeV deuterons and applied magnetic analysis to the α-particles emitted at right angles with respect to the deuteron beam. Unfortunately their spectrum of α-particles is not given in function of energy but in arbitrary units of the magnetic field. Based on the position of the peak indicating the 4.59 MeV ^7Li-level, we computed the magnetic field strength that would correspond to α-particles involved in a transition to the 5.5 MeV level.

([7]) J. M. BLAIR *et al.*, quoted by H. H. GOLDSMITH, H. W. IBSER and B. T. FELD: *Rev. Mod. Phys.*, **19**, 259 (1947).

([8]) P. STOLL and M. WÄCHTER: *Nuovo Cimento*, **10**, 347 (1953).

([9]) H. E. GOVE and J. A. HARVEY: *Phys. Rev.*, **82**, 658 (1951).

([10]) A. ASHMORE and J. F. RAFFLE: *Proc. Phys. Soc.*, **65** A, 297 (1952).

Taking into account the difference between laboratory and center-of-mass system of the compound nucleus, but neglecting the target-absorption correction, we obtained $H = 885 \pm 25$ arb. units for the 5.5 ± 0.3 MeV level. In fact, the experimental points taken by ASHMORE and RAFFLE exhibit a rise in the region of 860 arb. units, though these authors have drawn their curve neglecting this rise. Consequently, it does not fit within the limits of error.

The experiments of GELINAS, CLASS and HANNA [11] differ from those discussed above merely by the use of 0.47 resp. 0.48 MeV deuterons and of photographic plates as detectors. Their observations extend over the ^{7}Li-excitation energy range between 3.9 MeV and 5.2 MeV. Therefore the detection of a 5.5 MeV level would have been impossible.

In a preliminary note, GELINAS and HANNA [12] reported on their search for ^{7}Li-levels between 3.5 MeV and 5.6 MeV. As their observations were obscured by ^{7}Li^{+++} recoils, they did not find the 4.6 MeV level discovered by GOVE and HARVEY at the same time. Therefore, the absence of the 5.5 MeV level in this experiment is not surprising.

GOVE and HARVEY [9] made a search for excited levels of ^{7}Li by means of inelastically scattered protons, deuterons and α-particles of energies 7.9, 14. and 31 MeV respectively.

The proton spectrum covers the range from 0 to 5 MeV ^{7}Li-excitation energy, consequently no peak corresponding to the 5.5 MeV level could occur.

The α-spectrum at an angle of 20° with respect to the bombarding α-beam does not show any peak corresponding to a 5.5 MeV level, whereas at 50° the presence of such a peak cannot be excluded because of the insufficient resolution at angles greater than 30°. This can be understood if the α-particle, leaving the ^{7}Li-nucleus in the 5.5 MeV state, is emitted preferably in the direction at right angles to the beam. The same consideration may be applied to the results obtained by elastic scattering of deuterons on ^{7}Li and to the α-spectrum obtained with the reaction ^{9}Be(d, α)^{7}Li.

As far as we know, the above mentioned experiments are the only ones covering the range near 5.5 MeV ^{7}Li excitation energy.

The observed 6.8 MeV level could be the same as the 6.56 ± 0.12 MeV level found from experiments with inelastically scattered protons [13].

The absolute value of the ^{7}Li(γ, α)^{3}H cross section is calculated from the known ^{12}C(γ, 3α) cross section data [14], comparing the number of events corresponding to each reaction. Our value of the cross- section at $E_\gamma = 6.13$ MeV

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[11] R. W. GELINAS, C. M. CLASS and S. S. HANNA: *Phys. Rev.*, **83**, 1260 (1951).
[12] R. W. GELINAS and S. S. HANNA: *Phys. Rev.*, **82**, 298 (1951).
[13] W. FRANZEN and J. G. LIKELY: *Phys. Rev.*, **87**, 667 (1952).
[14] F. K. GOWARD and J. J. WILKINS: *Proc. Roy. Soc.*, **217**, 357 (1953).

is higher than the one found by irradiating with monochromatic γ-rays ([15]) because of overlapping of the two nearby levels at $E_\gamma = 5.5$ and $E_\gamma = 6.8$ MeV.

3. – Angular Distribution.

3·1. *Theoretical Analysis.* – For each possible value J of the spin of an excited level the angular correlation function $f(\theta)$ can be calculated, which measures the probability for the ^{3}H-particle to be emitted at an angle θ with respect to the direction of the incident γ-ray.

We assume that:

1) the $^7Li(\gamma, \alpha)^3H$ reaction has isolated resonances,

2) the excited levels have sufficiently short life-times.

Then, writing the reaction formula in the following form

$$\begin{array}{cccccc} \gamma & & -\ ^7\mathrm{Li} \rightarrow & ^7\mathrm{Li}^* \rightarrow \alpha + & & ^3\mathrm{H} \\ S_1 = 1 & J_A = \frac{3}{2} & J_B & S_2 = 0 & J_C = \frac{1}{2} & \end{array}$$

the theory of angular correlation ([16]) leads to

$$f(\theta) = \sum_k C^{k0}_{J_1-1J_11} C^{k0}_{J_20J_20} W(J_B J_1 J_B J_1;\ J_A k) \cdot W(J_B J_2 J_B J_2;\ J_C k) \cdot P_k(\cos\theta)\,.$$

The meaning of S_1, J_A, J_B, S_2, J_C is evident from the preceding formulation of the reaction, whereas

J_1 is the multipole order of radiation.

$\boldsymbol{J}_2 = \boldsymbol{L}_2 + \boldsymbol{S}_2$; L_2 denotes the angular momentum of the α-particle with respect to the ^{3}H-particle.

C denotes the Clebsch-Gordon, W the Racah coefficient, P_k the Legendre polynomial. The sum has to be extended over all positive, even integers 0, 2, 4,... such that

$$k \leqslant 2J_1,\ 2J_B,\ 2J_2\,.$$

Limiting the calculation to the cases of $E1$, $M1$ and $E2$ absorption and therefore $J_B < \frac{9}{2}$, the resulting functions $f(\theta)$ are shown in Table I and in Fig. 3.

([15]) H. Nabholz, P. Stoll and H. Wäffler: *Helv. Phys. Acta*, **25**, 701 (1952).
([16]) G. Racah: *Phys. Rev.*, **84**, 910 (1951).

TABLE I.

Multipole order of γ-ray	J_1	J_B	J_2	$f(\theta)$	Nr.
$E1$	1	5/2	2	$1 - 0.5 \cos^2 \theta$	1
	1	3/2	2	$1 + 0.75 \cos^2 \theta$	2
	1	1/2	0	const.	3
$M1$	1	5/2	3	$1 - 0.5 \cos^2 \theta$	1
	1	3/2	1	$1 + 0.75 \cos^2 \theta$	2
	1	1/2	1	const.	3
$E2$	2	7/2	3	$1 + 3.528 \cos^2 \theta - 2.647 \cos^4 \theta$	4
	2	5/2	3	$1 - 1.875 \cos^2 \theta + 2.5 \cos^4 \theta$	5
	2	3/2	1	const.	3
	2	1/2	1	const.	3

In order to determine J_B, these functions have to be compared with the experimental distributions $W(\theta)$. For this purpose the calculated functions have been normalized, so that (*).

$$\int_{20^0}^{160^0} f(\theta)\,\mathrm{d}\theta = \int_{20^0}^{160^0} W(\theta)\,\mathrm{d}\theta\ .$$

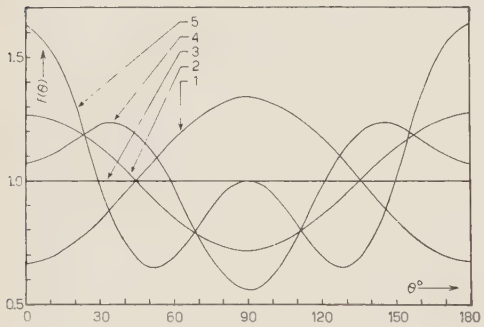

Fig. 3. – Calculated angular distribution of ^{3}H-particles from ^{7}Li(γ, α)^{3}H. (See also Table I).

The number of tracks has been multiplied by a correction factor, which depends on the track length (i.e. on E_γ) and on the angle θ.

Table II gives this correction factor ([17]).

The angular intervals 0°–20° and 160°–180°, for which the correction factor is very large, whilst the number of tracks is meager, have been excluded from the graphical representation.

(*) This normalization is equivalent to determining a constant factor by which we should multiply $f(\theta)$ in order to obtain the best fit to the experimental distribution $W(\theta)$. A calculation of this constant factor by the method of least squares is in agreement with our results.

([17]) M. WÄCHTER: *Diplomarbeit ETH* (1954).

TABLE II.

θ°	E^*(MeV)		
	4.7	5.5	6.8
0–20	5.34	5.37	5.41
20–40	1.87	1.90	1.94
40–60	1.40	1.42	1.48
60–80	1.65	1.67	1.73
80–100	1.69	1.72	1.78
100–120	1.65	1.67	1.73
120–140	1.40	1.42	1.48
140–160	1.87	1.90	1.94
160–180	5.34	5.37	5.41

3·2. *Experimental Results.* – The experimental distributions $W(\theta)$ are plotted in Fig. 4, together with the theoretical curves.

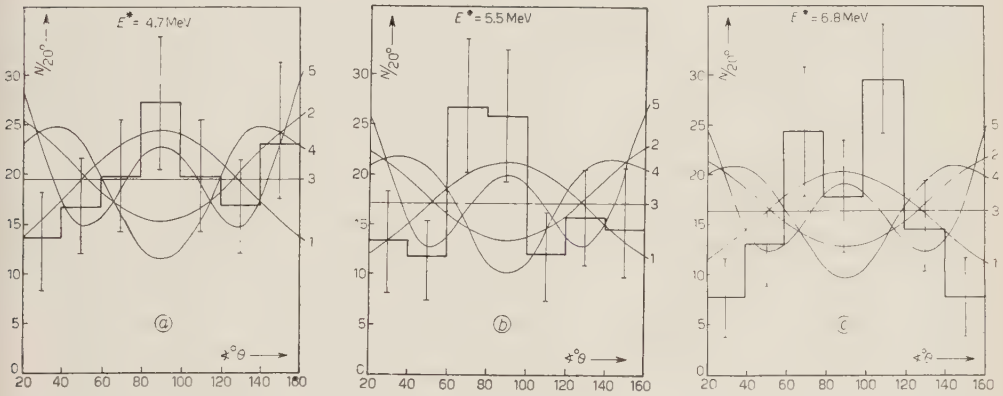

Fig. 4. – Track numbers vs. angle θ between the ^3H particle and the incident γ-ray direction. Figs. *a*, *b* and *c* refer to the 4.7 MeV, 5.5 MeV and 6.8 MeV levels in ^7Li, respectively. I = mean statistical error.

It can be seen that the statistical fluctuations of the track numbers are too great as to permit any definite conclusion concerning J_B. Nevertheless, some of the theoretical curves can be excluded. In particular, if we assume the probability of $E1$ γ-absorption in the energy range below 10 MeV to be small compared with the probability of $E2$ and $M1$ transitions ([18]), we can

([18]) J. M. BLATT and V. F. WEISSKOPF: *Theoretical Nuclear Physics* (New York, 1952), p. 656.

fix tentatively the parities as shown in Table III and the following may be pointed out:

I. $E^* = 4.7$ MeV.

The function 4 does not fit the measured distribution within the statistical error, thus the value $J_B = 7/2$ is excluded. $J_B = 3/2$ with $E1$ or $M1$ absorption is not probable (function 2). $J_B = 5/2$ (functions 1 and 5) is strongly indicated: $M1$, resp. $E2$ absorption.

II. $E^* = 5.5$ MeV.

For the same reasons as above, the experimental distribution points to $J_B = 5/2$: $M1$ absorption; $J_B = 3/2$ with E2 or $J_B = 1/2$ with $E2$ or $M1$

TABLE III.

E^*(MeV)		J_{Exp}		I [19]		II [20]		Multipole order of γ-ray
Our measurements	Other [21] measur.	Our measurements	Other [21] measurements	E^* (MeV)	J	E^* (MeV)	J	
—	0	—	$3/2^-$	0	3/2	0	3/2	—
—	0.478	—	$1/2^-$	0.478 *	1/2	0.35	1/2	—
4.7	4.67	$5/2^-$	—	4.6	7/2	4.02 4.48 5.16	5/2 3/2 5/2	— $E2$, $M1$
5.5	—	$5/2^-$, $3/2^-$, $1/2^-$ ($1/2^+$)	—	5.45	5/2	5.42	5/2	$E2$; $M1$ ($E1$)
—	6.56	—	($1/2^+$, $3/2^+$)	—	—	—	—	—
6.8	—	$5/2^-$, $3/2^-$, $1/2^-$ ($1/2^+$)	—	—	—	—	—	$E2$; $M1$
(7.4)	7.46	—	(5/2)	—	—	7.53	1/2	—
(8.3)	—	—	—	—	—	7.92	1/2	—
(9.0)	9.3 [22]	—	—	—	—	8.93	3/2	—

(*) By adjustment of a free parameter.

([19]) D. R. INGLIS: *Rev. Mod. Phys.*, **25**, 390 (1953).
([20]) R. SCHULTEN: *Zeits. f. Naturfor.*, **8***a*, 759 (1953).
([21]) F. AJZENBERG and T. LAURITSEN: *Rev. Mod. Phys.*, **24**, 321 (1952).
([22]) E. W. TITTERTON and T. A. BRINKLEY: *Proc. Phys. Soc.*, **66**, 194 (1953).

absorption cannot be excluded (function 3), whereas $J_B = 3/2$ with $M1$, $J_B = 7/2$, with $E2$ absorption may be excluded.

III. $E^* = 6.8$ MeV.

The same conclusion as for $E^* = 5.5$ MeV; moreover $J_B = 5/2$ with $E2$ absorption (function 5) may be excluded.

A summary of the results is given in Table III. For comparison, the values based on theoretical calculations are included ([19,20]).

4. – General Remarks.

A fact which seems very interesting from the theoretical point of view is the following.

The number of levels found experimentally in ^7Li is less than what one would expect theoretically ([17,18]). Particularly in the low energy region of ^7Li only few levels are known, and it might be interesting to investigate whether some kind of selection rule exists which limits the number of possible states in this region.

The shell model allows for the three nucleons in the p-shell a well-defined number of states: 21. This number is independent of the coupling scheme used in the calculation. The separation of the levels and in particular the number of low energy levels depend on the type and strength of the interaction assumed, and is particularly sensitive to the ratio R_p/R_r of the range of the potential to that of the eigenfunctions.

We wish to point out that the small number of low energy levels found in ^7Li might be interpreted as indicating that the spatial symmetry of the nucleon configuration plays an important role in determining the position of the levels.

It is well known that the states symmetric with respect to the exchange of the spatial coordinates of any two nucleons have a lower kinetic energy and a potential energy higher in absolute value than those antisymmetric in space. Therefore, assuming that the low energy states are only those space symmetric (*) the number of low energy levels is greatly reduced. For example in ^7Li, of the 21 states allowed by the shell model, only 4 are symmetric in space (see Appendix). This assumption is supported by the fact that the calculations with an L-S interaction predict that in ^7Li the lower states are the

(*) Here and henceforth space symmetry is meant with respect to the symmetric permutation group.

doublets ^2P and ^2F, and in ^{6}Li the states ^3S, ^3D, ^1S, ^1D which are all symmetric in space.

Only in the case of an L-S interaction model can one speak of pure space symmetries. Both the j-j and the intermediate interactions mix the states of different spatial symmetries and tend, therefore, to lift the levels with respect to the L-S interaction calculation.

The qualitative character of these considerations does not allow one to draw any definite conclusion about the interaction model. For example one cannot exclude an α-particle structure of ^{7}Li, because the ground state of ^{3}H is also space symmetric. One might therefore think of the excited states of ^{7}Li as different states of orbital angular momentum of ^{3}H with respect to the α-core.

We think that it would be highly desirable to make a systematic search for levels in ^{7}Li, particularly in the low energy region, in order to check the possibility that some levels may not have been observed.

5. – Appendix.

Both with Slater's method and the group theoretical formula, one can find the number of states with angular momentum J of a system of n equivalent nucleons having a certain spatial symmetry with respect to the symmetric permutation group. Assuming an L-S coupling so that one can define states of pure spatial symmetry with respect to the symmetric permutation group, the 4 space symmetric states of the p^3 configuration of ^{7}Li are: (3, 7/2, 1/2), (3, 5/2, 1/2), (1, 3/2, 1/2), (1, 1/2, 1/2), where the numbers in brackets are, in order, L, J, T.

6. – Acknowledgements.

We are greatly indebted to Professor P. SCHERRER for his kind interest and advice, and to Professor M. VERDE for stimulating discussions.

RIASSUNTO

Si studia la reazione ^{7}Li(γ, α)^{3}H mediante emulsioni nucleari contenenti Li esposte alla radiazione di un betatrone di 31 MeV. È stata misurata la sezione d'urto in funzione dell'energia dei γ e sono stati trovati per il ^{7}Li i livelli energetici seguenti: 4.7, 5.5, 6.8, 8.3, 9 MeV. I tre primi livelli hanno una intensità relativa di 1 : 0.75 : 0.75. L'analisi della distribuzione angolare permette d'escludere certi valori possibili dello spin per i livelli di 4.7, 5.5, 6.8 MeV. Il valore assoluto della sezione d'urto è ottenuto per confronto con il valore corrispondente della reazione ^{12}C(γ, 3α). Si discute una regola di selezione che limita il numero dei livelli energetici per le basse energie.

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A Non-Linear Generalization of the Schrödinger and Dirac Equations (II).

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Summary. — The method of generalization applied in I to the Schrödinger and Dirac wave equations and to the classical Hamilton-Jacobi equation is extended to all the theories derived from variational principles admitting the gauge invariance of the first kind. By a slight change of the ordinary definition of the gauge transformations of the first kind, the Hamilton-Jacobi theory and the dynamics of the inviscid barotropic fluids are also included. In all cases, the method of generalization corresponds to the passage from the irrotational to the vortex motions of the inviscid fluids, although the part of the vorticity be taken over by other quantities. It is shown that theorems analogous to those of Helmholtz, Kelvin and Cartan and the Cauchy formula of the theory of the vortex motions of an inviscid barotropic fluid can be extended to the generalization of the gauge invariant field formalism, the vorticity being replaced by a non-integrability vector appearing in the formalism. The generalization method may be regarded as consisting in the replacement of quantities similar to the phases of the wave functions by non integrable quantities.

1. – Introduction.

In the first part of this paper [1] we have discussed a method of associating a non linear system of equations to the linear wave equations for charged particles. We examined only the Schrödinger and Dirac equations, but the method can be applied obviously to the Pauli, Klein-Gordon and Proca equations in a straightforward way, since it amounts to add to the electromagnetic potentials A^0, $\boldsymbol{A}$, in the lagrangian, the four vector B^0, $\boldsymbol{B}$

$$B^0 = \lambda \frac{\partial \mu}{\partial t}, \qquad \boldsymbol{B} = -\lambda \frac{\partial \mu}{\partial \boldsymbol{x}}, \tag{1}$$

[1] M. Schönberg: *Nuovo Cimento*, **11**, 674 (1954). This paper will be denoted by I.

λ and μ being variated independently in the variational principle. The same procedure can also be applied to the non relativistic Hamilton-Jacobi equation, as shown in I, and it is seen to correspond to the introduction of potentials which give rise to a vanishing Lorentz force in the motions in consideration.

TAKABAYASI ([2]) and the author ([3]) have found independently that a generalization of the Schrödinger equation is suggested by the Madelung hydrodynamical model. This generalization coincides with that obtained by the method of I, in the case of the Schrödinger equation, but the extension of the hydrodynamical generalization to other cases is not always easy and seems rather artificial, even in the case of the Klein-Gordon equation discussed by TAKABAYASI ([2]), whereas the method of I can be easily applied in all cases and is related to the fundamental property of gauge-invariance.

In this paper we shall discuss in detail the relations between the method given in I and a generalization of the gauge transformations. We shall examine the case of a gauge-invariant field theory whose lagrangian density L is a real function of the complex field quantities $U^{(r)}, U^{*(r)}$ and their first order derivatives with respect to the space and time coordinates

$$U_l^{(r)} = \frac{\partial U^{(r)}}{\partial x_l}, \qquad U_l^{*(r)} = \frac{\partial U^{*(r)}}{\partial x_l}, \tag{2}$$

$$x_0 = t, \qquad x_1, x_2, x_3 = \text{cartesian coordinates}. \tag{2a}$$

We shall not assume that L is relativistically invariant and the gauge-invariance will be assumed only for the particular transformations

$$U^{(r)} \to e^{i\alpha} U^{(r)}, \qquad U^{*(r)} \to e^{-i\alpha} U^{*(r)}, \tag{3}$$

$$\alpha = \text{arbitrary real constant}. \tag{3a}$$

The invariance of L for the transformations (3), with an arbitrary α, gives the equation

$$\sum_r \left\{ \left(U^{(r)} \frac{\partial L}{\partial U^{(r)}} + \sum_{l=0}^{3} U_l^{(r)} \frac{\partial L}{\partial U_l^{(r)}} \right) - \left(U^{*(r)} \frac{\partial L}{\partial U^{*(r)}} + \sum_{l=0}^{3} U^{*(r)} \frac{\partial L}{\partial U_l^{*(r)}} \right) \right\} = 0. \tag{4}$$

It follows from (4) and the field equations

$$\frac{\partial L}{\partial U^{(r)}} - \sum_{l=0}^{3} \frac{\partial}{\partial x_l} \frac{\partial L}{\partial U_l^{(r)}} = 0, \qquad \frac{\partial L}{\partial U^{*(r)}} - \sum_{l=0}^{3} \frac{\partial}{\partial x_l} \frac{\partial L}{\partial U_l^{*(r)}} = 0, \tag{5}$$

([2]) T. TAKABAYASI: *Prog. Theor. Phys.*, **9**, 187 (1953).
([3]) M. SCHÖNBERG: *Nuovo Cimento*, **12**, 103 (1954).

that there is a continuity equation (4)

$$\sum_{l=0}^{3} \frac{\partial j_l}{\partial x_l} = 0 . \tag{6}$$

$$j_l = - i\varepsilon \sum_r \left(\frac{\partial L}{\partial U_l^{(r)}} U^{(r)} - U^{*(r)} \frac{\partial L}{\partial U_l^{(r)*}} \right) \tag{7}$$

$$\varepsilon = \text{real constant.} \tag{7a}$$

j_0 is the charge density and $\boldsymbol{j}$ the current density of the field, with a suitable choice of the constant ε. In order to extend the method of generalization given in I to the present case, we shall replace L by a new lagrangian $\underline{L}$ obtained from L by replacing the $U_l^{(r)}$, $U_l^{*(r)}$ by the $\underline{U}_l^{(r)}$, $\underline{U}_l^{*(r)}$,

$$\begin{cases} \underline{U}_l^{(r)} = d_l U^{(r)} = U_l^{(r)} + \dfrac{i}{\hbar} \lambda \dfrac{\partial \mu}{\partial x_l} U^{(r)} , \\ \underline{U}_l^{*(r)} = d_l^* U^{*(r)} = U_l^{*(r)} - \dfrac{i}{\hbar} \lambda \dfrac{\partial \mu}{dx_l} U^{*(r)} , \end{cases} \tag{8}$$

$$\lambda, \mu = \text{real functions} . \tag{8a}$$

We shall now take the following variational principle

$$\delta \int \underline{L} \, dt \, d\boldsymbol{x} = 0 , \tag{9}$$

in which the $U^{(r)}$, $U^{*(r)}$ and λ, μ are variated independently. The Euler Equations of (9) are

$$\sum_{l=0}^{3} d_l^* \frac{\partial \underline{L}}{\partial \underline{U}_l^{(r)}} - \frac{\partial \underline{L}}{\partial U^{(r)}} = 0 , \qquad \sum_{l=0}^{3} d_l \frac{\partial \underline{L}}{\partial \underline{U}_l^{*(r)}} - \frac{\partial \underline{L}}{\partial U^{*(r)}} = 0 , \tag{10}$$

$$\sum_{l=0}^{3} \frac{\partial \mu}{\partial x_l} \sum_r \left(\frac{\partial \underline{L}}{\partial \underline{U}_l^{(r)}} U^{(r)} - \frac{\partial \underline{L}}{\partial \underline{U}_l^{*(r)}} U^{*(r)} \right) = 0 , \tag{11}$$

$$\sum_{l=0}^{3} \frac{\partial}{\partial x_l} \left\{ \lambda \sum_r \left(\frac{\partial \underline{L}}{\partial \underline{U}_l^{(r)}} U^{(r)} - \frac{\partial \underline{L}}{\partial \underline{U}_l^{*(r)}} U^{*(r)} \right) \right\} = 0 . \tag{12}$$

By taking for the charge and current density

$$j_l = - i\varepsilon \sum_r \left(\frac{\partial \underline{L}}{\partial \underline{U}_l^{(r)}} \underline{U}^{(r)} - U^{*(r)} \frac{\partial \underline{L}}{\partial \underline{U}_l^{*(r)}} \right) , \tag{13}$$

(4) W. Pauli: *Rev. Mod. Phys.*, **13**, 203 (1941).

there is a continuity equation

$$\sum_{l=0}^{3} \frac{\partial j_l}{\partial x_l} = 0 , \tag{14}$$

which follows from (10) and (4). By taking into account (13) and (14), we may write the equations for λ and μ as follows

$$\sum_{l=0}^{3} j_l \frac{\partial \lambda}{\partial x_l} = 0 , \qquad \sum_{l=0}^{3} j_l \frac{\partial \mu}{\partial x_l} = 0 . \tag{15}$$

We have also the continuity equations

$$\sum_{l=0}^{3} \frac{\partial}{\partial x_l} (\lambda j_l) = 0 , \qquad \sum_{l=0}^{3} \frac{\partial}{\partial x_l} (\mu j_l) = 0 . \tag{16}$$

It is interesting to notice that any function $f(\lambda, \mu)$ satisfies the equations

$$\sum_{l=0}^{3} j_l \frac{\partial f}{\partial x_l} = 0 , \qquad \sum_{l=0}^{3} \frac{\partial}{\partial x_l} (f j_l) = 0 . \tag{17}$$

Therefore

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} f j_0 \, \mathrm{d}\boldsymbol{x} = 0 , \tag{18}$$

when the $f j_l$ tend to zero at infinity fast enough.

The above formalism contains as particular cases the non linear generalizations of the Schrödinger and Dirac equations given in I. With any solution of equations (5) we can build solutions of the system (10)–(15), since, by taking λ and μ constant, the equations (15) are satisfied and the equations (10) go over into (5). *We may say that all the solutions of* (5) *are also solutions of* (10)–(15).

L is invariant for a much larger group of gauge-transformations than L. Let α be an arbitrary real function of λ and μ. The differential expression $\lambda \, \mathrm{d}\mu - \hbar \, \mathrm{d}\alpha$ admits an integrating factor $1/\lambda'$

$$\frac{1}{\lambda'} (\mathrm{d}\mu - \hbar \, \mathrm{d}\alpha) = \mathrm{d}\mu' . \tag{19}$$

The transformation

$$\begin{cases} U^{(r)} \to e^{i\alpha} U^{(r)} & U^{*(r)} \to e^{-i\alpha} U^{*(r)}, \\ \lambda \to \lambda' , & \mu \to \mu' , \end{cases} \tag{20}$$

leads to

$$\underline{U}_l^{(r)} \to e^{i\alpha}\underline{U}_l^{(r)}, \qquad \underline{U}_l^{*(r)} \to e^{-i\alpha}\underline{U}_l^{*(r)}, \tag{21}$$

so that the new $\underline{L}$ is equal to the old one. The transformation (20) leads also to a new solution of the system (10)–(15) when applied to a solution $U^{(r)}$, λ, μ of (10)–(15). Indeed, it follows from (20)–(21) that the j_l are invariant, thus λ', μ' being functions of λ, μ do satisfy the equations (15). On the other hand

$$d_l' = d_l - i\frac{\partial\alpha}{\partial x_l}, \qquad d_l'^* = d_l^* + i\frac{\partial\alpha}{\partial x_l}, \tag{22}$$

therefore the new $U^{(r)}$ and $U^{*(r)}$ satisfy (10). Thus we get the theorem:

The transformations defined by (19) *and* (20) *do not change* L *and the* j_l *and replace a solution* U, U^*, λ, μ *of the system* (10)–(15) *by another solution* U', U'^*, λ', μ' *of the same system.*

It follows from (19) that it is possible to take λ' and μ' as constants, when λ is a function of μ, by choosing $\alpha = \hbar^{-1}\int\lambda\,\mathrm{d}\mu$. In this case the U', U'^*, are solutions of (5). When λ and μ are independent, it is not possible to have a constant μ', with any choice of α. The solutions of (10)–(15) such that λ be a function of μ may be considered as trivial generalizations of those of (5), since they can be obtained from those of (5) by a gauge transformation of the first kind. This is no more possible when λ and μ are independent.

The above formalism seems to depend essentially on the Planck constant h. It is however possible to present the formalism of a gauge invariant field theory in a different and more general way, applicable also to the classical Hamilton-Jacobi theory. Instead of a density of lagrangian L depending on pairs of complex conjugate functions, U, U^*, we may consider L as a function of couples of real functions $R^{(r)}$ and $S^{(r)}$. Whenever L depends on the U, U^*, we can take

$$U^{(r)} = R^{(r)}\exp\left(\frac{i}{\hbar}S^{(r)}\right), \qquad U^{*(r)} = R^{(r)}\exp\left(-\frac{i}{\hbar}S^{(r)}\right) \tag{23}$$

L can be obtained by replacing in L the derivatives $\partial S^{(r)}/\partial x_l$ by $\partial S^{(r)}/\partial x_l + \lambda\partial\mu/\partial x_l$).

The special gauge-transformation (3)–(3a) corresponds to the transformation

$$S^{(r)} \to S^{(r)} + \hbar\alpha = S^{(r)} + \beta \qquad (\beta = \text{constant}). \tag{24}$$

Equation (4) is replaced now by

$$\sum_r \frac{\partial L}{\partial S^{(r)}} = 0 \tag{25}$$

and (7) by

$$j_l = -\varepsilon' \sum_r \frac{\partial L}{\partial S_l^{(r)}}, \qquad \left(S_l^{(r)} = \frac{\partial S^{(r)}}{\partial x_l}\right), \tag{26}$$

$$\varepsilon' = \text{constant}. \tag{26a}$$

In the generalized formalism we have

$$j_l = -\varepsilon' \sum_r \frac{\partial L}{\partial \underline{S}_l^{(r)}}, \qquad \left(\underline{S}_l^{(r)} = S_l^{(r)} + \lambda \frac{\partial \mu}{\partial x_l}\right). \tag{27}$$

2. – The stress tensors.

It is well known that the interaction with the electromagnetic field can be introduced by replacing in L the $S_l^{(r)}$ by the $\bar{S}_l^{(r)}$

$$\bar{S}_l^{(r)} = S_l^{(r)} - \frac{\varepsilon'}{c} A_l, \tag{28}$$

A_1, A_2 and A_3 being the components of the vector potential $\boldsymbol{A}$ and A_0 the scalar potential A^0 multiplied by c. Thus the generalization of the ordinary formalism by the introduction of the $\underline{S}_l^{(r)}$ may be regarded as due to the introduction of an interaction with an electromagnetic field described by the potentials B_l

$$\frac{\varepsilon'}{c} B_l = -\lambda \frac{\partial \mu}{\partial x_l}. \tag{29}$$

It follows from (29) that

$$\frac{\varepsilon'}{c} \sum_{l=0}^{3} \left(\frac{\partial B_l}{\partial x_k} - \frac{\partial B_k}{\partial x_l}\right) j_l = \sum_{l=0}^{3} \left(j_l \frac{\partial \lambda}{\partial x_l}\right) \frac{\partial \mu}{\partial x_k} - \sum_{l=0}^{3} \left(j_l \frac{\partial \mu}{\partial x_l}\right) \frac{\partial \lambda}{\partial x_k}, \tag{30}$$

and by taking into account equations (15) we get

$$\sum_{l=0}^{3} \left(\frac{\partial B_l}{\partial x_k} - \frac{\partial B_k}{\partial x_l}\right) j_l = 0. \tag{31}$$

The density of Lorentz force exerted by the field of potentials B_l on the charge and current distribution j_l is nil.

The components of the stress tensor of the field corresponding to L are

$$T_{kl} = \sum_r \left(R_k^{(r)} \frac{\partial L}{\partial R_l^{(r)}} + S_k^{(r)} \frac{\partial L}{\partial S_l^{(r)}}\right) - \delta_{kl} L. \tag{32}$$

When there is an electromagnetic field we must take

$$\overline{T}_{kl} = \sum_r \left(R_k^{(r)} \frac{\partial L}{\partial R_l^{(r)}} + \overline{S}_l^{(r)} \frac{\partial L}{\partial \overline{S}_l^{(r)}} \right) - \delta_{kl} L \,, \tag{33}$$

L being obtained from L by replacing the $S_l^{(r)}$ by the $\overline{S}_l^{(r)}$, in order to have

$$\sum_{l=0}^{3} \frac{\partial}{\partial x_l} T_{kl} = \frac{1}{c} \sum_{l=0}^{3} j_l \left(\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l} \right) \qquad \left(j_l = - \varepsilon' \sum_r \frac{\partial \overline{L}}{\partial \overline{S}_l^{(r)}} \right). \tag{34}$$

When there is no electromagnetic field, we have simply

$$\sum_{l=0}^{3} \frac{\partial}{\partial x_l} T_{kl} = 0 \,. \tag{35}$$

In the generalized formalism λ and μ are also field variables, so that the standard rule gives

$$\underline{T}_{kl} = \sum_r \left(R_k^{(r)} \frac{\partial L}{\partial R_l^{(r)}} + S_k^{(r)} \frac{\partial L}{\partial S_l^{(r)}} - \frac{\partial \mu}{\partial x_k} \frac{\partial L}{\partial (\partial \mu / \partial x_l)} \right) - \delta_{kl} L = \tag{36}$$

$$= \sum_r \left(R_l^{(r)} \frac{\partial L}{\partial R_k^{(r)}} + S_k^{(r)} \frac{\partial L}{\partial S_l^{(r)}} \right) - \delta_{kl} L \,.$$

We must, of course, have

$$\sum_{l=0}^{3} \frac{\partial}{\partial x_l} \underline{T}_{kl} = 0 \,, \tag{37}$$

since there is no true interaction with the electromagnetic field. On the other hand the form of T_{kl} is the same as if there would be an external field described by the potential B_l. In order that (34) and (37) be compatible, the Lorentz forces corresponding to the B_l must be nil, as happens indeed.

There is no difficulty in introducing the interaction with the electromagnetic field in the generalized formalism. It suffices to replace in $\underline{L}$ the $S_l^{(r)}$ by the $\underline{\overline{S}}_l^{(r)}$

$$\underline{\overline{S}}_l^{(r)} = \underline{S}_l^{(r)} - \frac{\varepsilon'}{e} A_l = S_l^{(r)} + \lambda \frac{\partial \mu}{\partial x_l} - \frac{\varepsilon'}{e} A_l \tag{38}$$

and to take the function $\underline{\overline{L}}$ thus obtained as density of lagrangian. The j_l must now be taken as follows

$$j_l = - \varepsilon' \sum_r \frac{\partial \underline{\overline{L}}}{\partial \underline{\overline{S}}_l^{(r)}} \,. \tag{39}$$

The components of the stress tensor are now

$$\underline{\bar{T}}_{kl} = \sum_r \left(R_k^{(r)} \frac{\partial \bar{L}}{\partial R_l^{(r)}} + \underline{\bar{S}}_k^{(r)} \frac{\partial \underline{\bar{L}}}{\partial \underline{\bar{S}}_l^{(r)}} \right) - \delta_{kl} \underline{\bar{L}} \,. \tag{40}$$

3. – Potentials giving rise to null Lorentz forces.

We shall now determine the most general potentials B_l which give rise to a vanishing density of Lorentz force for a given charge and current distribution j_l. It is known from the theory of the Pfaff expressions that it is always possible to find two pairs of functions λ_1, μ_1 and λ_2, μ_2 such that

$$\sum_{l=0}^{3} B_l \, \mathrm{d}x_l = \lambda_1 \, \mathrm{d}\mu_1 + \lambda_2 \, \mathrm{d}\mu_2 \,. \tag{41}$$

It follows from (41) that

$$B_l = \lambda_1 \frac{\partial \mu_1}{\partial x_l} + \lambda_2 \frac{\partial \mu_2}{\partial x_l} \,. \tag{42}$$

Hence

$$\frac{\partial B_l}{\partial x_k} - \frac{\partial B_k}{\partial x_l} = \left(\frac{\partial \lambda_1}{\partial x_k} \frac{\partial \mu_1}{\partial x_l} - \frac{\partial \lambda_1}{\partial x_l} \frac{\partial \mu_1}{\partial x_k} \right) + \left(\frac{\partial \lambda_2}{\partial x_k} \frac{\partial \mu_2}{\partial x_l} - \frac{\partial \lambda_2}{\partial x_l} \frac{\partial \mu_2}{\partial x_k} \right) .$$

The conditions for the Lorentz force to vanish are

$$\sum_{l=0}^{3} j_l \left(\frac{\partial B_l}{\partial x_k} - \frac{\partial B_k}{\partial x_l} \right) = \frac{\partial \lambda_1}{\partial x_k} \sum_{l=0}^{3} j_l \frac{\partial \mu_1}{\partial x_l} - \frac{\partial \mu_1}{\partial x_k} \sum_{l=0}^{3} j_l \frac{\partial \lambda_1}{\partial x_l} + \tag{43}$$

$$+ \frac{\partial \lambda_2}{\partial x_k} \sum_{l=0}^{3} j_l \frac{\partial \mu_2}{\partial x_l} - \frac{\partial \mu_2}{\partial x_k} \sum_{l=0}^{3} j_l \frac{\partial \lambda_2}{\partial x_l} = 0 \,.$$

Equations (43) may be regarded as a system of linear equations with four unknown

$$\alpha_s = \sum_{l=0}^{3} j_l \frac{\partial \lambda_s}{\partial x_l} \,, \qquad \beta_s = \sum_{l=0}^{3} j_l \frac{\partial \mu_s}{\partial x_l} \qquad (s = 1, 2) \,. \tag{44}$$

If the four functions λ_s, μ_s would be independent, the functional determinant

$$\Delta = \frac{D(\lambda_1, \mu_1, \lambda_2, \mu_2)}{D(x_0, x_1, x_2, x_3)} \,, \tag{45}$$

would be non null in some region of the space-time. In this case the solutions

α_s, β_s of (43) would vanish in that region, so that the j_l would satisfy the equations

$$\sum_{l=0}^{3} j_l \frac{\partial \lambda_s}{\partial x_l} = 0 , \qquad \sum_{l=0}^{3} j_l \frac{\partial \mu_s}{\partial x_l} = 0 , \tag{46}$$

whose determinant is again Δ. The only solution of (46) would be $j_0 = j_1 = j_2 = j_3 = 0$. Therefore, excluding the trivial case of null j's, we must have

$$\Delta = 0 , \tag{47}$$

in order that the Lorentz force be nil.

Given the potentials B_l satisfying the condition (47), it is possible to find non vanishing j_l that are solutions of (46) and for such j_l

$$\sum_{l=0}^{3} j_l \left(\frac{\partial B_l}{\partial x_k} - \frac{\partial B_k}{\partial x_l} \right) = 0 . \tag{48}$$

The minimum number of independent functions of the variables y in terms of which a differential form $\sum_r Y_r \mathrm{d}y_r$ can be expressed is called the class of the form. The above discussion shows that:

A necessary and sufficient condition for the existence of a non null charge and current distribution j_l for which the Lorentz forces derived from the potentials B_l be nil is that the class of the Pfaff expressions $\sum_{l=0}^{3} B_l \mathrm{d}x_l$ be less than 4.

The determination of the class of the differential form $\sum_{l=0}^{3} B_l \mathrm{d}x_l$ does not require the knowledge of a canonical decomposition (41). It is shown in the theory of the Pfaff equations that the class of $\sum_{l=0}^{3} B_l \mathrm{d}x_l$ is the number of independent equations of the sistem

$$\sum_{l=0}^{3} \left(\frac{\partial B_l}{\partial x_k} - \frac{\partial B_k}{\partial x_l} \right) \mathrm{d}x_l = 0 , \qquad \sum_{l=0}^{3} B_l \mathrm{d}x_l = 0 , \tag{49}$$

i.e. the rank of its matrix when it is viewed as a system of linear equations in the variables $\mathrm{d}x_l$.

Any Pfaff expression $\sum_{l=0}^{3} B_l \mathrm{d}x_l$, of class less than 4, can be written as

$$\sum_{l=0}^{3} B_l \mathrm{d}x_l = \mathrm{d}\varphi + \lambda \mathrm{d}\mu . \tag{50}$$

The term $\mathrm{d}\varphi$ is irrelevant for the Lorentz force. It corresponds to an ordinary change of the gauge of the potentials. We shall now determine the most general form of the Lorentz force which vanishes for a given motion of a

charge distribution defined by the function j_l. We have

$$\sum_{l=0}^{3}\left(\frac{\partial B_l}{\partial x_k}-\frac{\partial B_k}{\partial x_l}\right)j_l=\frac{\partial\lambda}{\partial x_k}\sum_{l=0}^{3}j_l\frac{\partial\mu}{\partial x_l}-\frac{\partial\mu}{\partial x_k}\sum_{l=0}^{3}j_l\frac{\partial\lambda}{\partial x_l}, \tag{51}$$

so that when the four vectors $\partial\lambda/\partial x_k$ and $\partial\mu/\partial x_k$ are not parallel, i.e. when λ and μ are independent, the condition $\sum_{l=0}^{3}[(\partial B_l/\partial x_k)-(\partial B_k/\partial x_l)]j_l=0$ requires that

$$\sum_{l=0}^{3}j_l\frac{\partial\lambda}{\partial x_l}=0\,,\qquad\sum_{l=0}^{3}j_l\frac{\partial\mu}{\partial x_l}=0\,. \tag{52}$$

When λ is a function of μ, $(\partial B_l/\partial x_k)-(\partial B_k/\partial x_l)=0$ and the Lorentz force vanishes for any motion of the charges. Summing up:

The most general velocity dependent force derived from potential B_l, which vanishes for a given motion of the charges defined by the j_l, can be obtained by taking the potentials $B_l=\lambda(\partial\mu/\partial x_l)$, λ and μ being two solutions of the equation $\sum_{l=0}^{3}j_l(\partial f/\partial x_l)=0$, i.e. two constants of the motion of the charge distribution.

The above theorem shows that our generalization of the equations of the gauge invariant charged field amounts to introduce the most general Lorentz force vanishing for the motions of the charge of the field. It is of course possible to replace the $S_l^{(r)}$ by $S_l^{(r)}+\sum_{s=1}^{n}\lambda_s(\partial\mu_s/\partial x_l)$, and this may even be convenient in some cases, but this will not allow for a more general type of vanishing Lorentz forces. One of the generalizations of the Dirac equation given in I involves essentially four pairs of Clebsch parameters. When several pairs of Clebsch parameters are introduced, the equations (15) are replaced by the following ones

$$\sum_{l=0}^{3}j_l\frac{\partial\lambda_s}{\partial x_l}=0\,,\qquad\sum_{l=0}^{3}j_l\frac{\partial\mu_s}{\partial x_l}=0\,.\qquad(s=1,2,\ldots,n). \tag{53}$$

4. – Non-integrable quantities and Clebsch parameters.

The generalized lagrangian $\underline{L}$ was derived from L by the substitution

$$\frac{\partial S^{(r)}}{\partial x_l}\to\frac{\partial S^{(r)}}{\partial x_l}+\lambda\frac{\partial\mu}{\partial x_l}\,. \tag{54}$$

In hydrodynamics, the passage from the irrotational motions, in which the velocity $\boldsymbol{v}$ is the gradient of a scalar Φ

$$\boldsymbol{v}=\frac{\partial\Phi}{\partial\boldsymbol{x}}\,, \tag{55}$$

to the general vortex motions can be done by introducing the Clebsch parameters Φ, λ, μ

$$\boldsymbol{v} = \frac{\partial \Phi}{\partial \boldsymbol{x}} + \lambda \frac{\partial \mu}{\partial \boldsymbol{x}} . \tag{56}$$

The equations of the irrotational motions of an inviscid fluid, whose pressure p is a function of the density ϱ (barotropic fluid), can be derived from a variational principle with

$$L = \varrho \left\{ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left(\frac{\partial \Phi}{\partial \boldsymbol{x}} \right)^2 + W + F(\varrho) \right\} , \tag{57}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varrho} (\varrho F) = \int \frac{\mathrm{d}p}{\varrho} , \tag{58}$$

W being the potential of the external force per unit mass. The equations for the vortex motions follow from the Bateman [5] variational principle, whose lagrangian density is

$$\underline{L} = \varrho \left\{ \frac{\partial \Phi}{\partial t} + \lambda \frac{\partial \mu}{\partial t} + \frac{1}{2} \left(\frac{\partial \Phi}{\partial \boldsymbol{x}} + \lambda \frac{\partial \mu}{\partial \boldsymbol{x}} \right)^2 + W + F(\varrho) \right\} . \tag{59}$$

The extension of the theory of the irrotational motions by the inclusion of the vorticity is a particular case of the generalization discussed in section **1**, as can be immediately seen by taking

$$S = \Phi , \qquad R = \sqrt{\varrho} . \tag{60}$$

The gauge invariance condition (25) is obviously satisfied. It is now related to the conservation of mass, instead of the conservation of charge.

The vorticity of the general motions of the inviscid fluid excludes the existence of a velocity potential. We may however say that there is a non integrable quantity Ω, which does not have definite values at the points of the fluid in a vortex region but has definite values in the irrotational regions, where it coincides with the velocity potential. The difference of the values of Ω at two neighbouring points $\boldsymbol{x}$ and $\boldsymbol{x}+\delta\boldsymbol{x}$, at the same time, is definite

$$\delta\Omega = \boldsymbol{v} \cdot \delta\boldsymbol{x} = \left(\frac{\partial \Phi}{\partial \boldsymbol{x}} + \lambda \frac{\partial \mu}{\partial \boldsymbol{x}} \right) \cdot \delta\boldsymbol{x} = \delta\Phi + \lambda \, \delta\mu . \tag{61}$$

We shall extend the definition of the difference of the values of Ω to the case

[5] H. Bateman: *Partial differential equations of mathematical physics* (New York, 1944), p. 164.

of two neighbouring points in space at two neighbouring times

$$\mathrm{d}\Omega = \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} - \left(\frac{\boldsymbol{v}^2}{2} + W + \int\frac{\mathrm{d}p}{\varrho}\right)\mathrm{d}t\,. \tag{62}$$

It follows from the Bateman variational principle that

$$\frac{\partial\Phi}{\partial t} + \lambda\frac{\partial\mu}{\partial t} + \frac{1}{2}\left(\frac{\partial\Phi}{\partial\boldsymbol{x}} + \lambda\frac{\partial\mu}{\partial\boldsymbol{x}}\right)^2 + W + \int\frac{\mathrm{d}p}{\varrho} = 0\,. \tag{63}$$

Hence

$$\mathrm{d}\Omega = \mathrm{d}\Phi + \lambda\mathrm{d}\mu\,. \tag{64}$$

It follows from (64) that the class of the Pfaff form $\mathrm{d}\Omega$ is less than 4, in any motion of the fluid. In the irrotational motions, the class of $\mathrm{d}\Omega$ is 1, since we can take $\lambda = \mu = 0$, in this case.

The condition that $\mathrm{d}\Omega$ *be integrable, or of class* 1, *gives the equation of the irrotational motions, except the continuity equation.* Indeed, when $\mathrm{d}\Omega$ is integrable, there is a function S such that

$$\frac{\partial S}{\partial\boldsymbol{x}} = \boldsymbol{v}\,, \qquad \frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial\boldsymbol{x}}\right)^2 + W + \int\frac{\mathrm{d}p}{\varrho} = 0\,. \tag{65}$$

The condition that $\mathrm{d}\Omega$ be of class lass than 4 is not sufficient to determine the equations of the trajectories of the elements of the fluid. Indeed, it follows from the canonical form of a Pfaff expression that there are three functions Φ, λ and μ such that (64) is satisfied. Hence

$$\boldsymbol{v} = \frac{\partial\Phi}{\partial\boldsymbol{x}} + \lambda\frac{\partial\mu}{\partial\boldsymbol{x}}\,, \qquad \frac{\partial\Phi}{\partial t} + \lambda\frac{\partial\mu}{\partial t} + \frac{1}{2}\left(\frac{\partial\Phi}{\partial\boldsymbol{x}} + \lambda\frac{\partial\mu}{\partial\boldsymbol{x}}\right)^2 + W + \int\frac{\mathrm{d}p}{\varrho} = 0\,. \tag{66}$$

In order to get the Euler equations, we must also have the equations

$$\frac{\partial\lambda}{\partial t} + \boldsymbol{v}\cdot\frac{\partial\lambda}{\partial\boldsymbol{x}} = 0\,, \qquad \frac{\partial\mu}{\partial t} + \boldsymbol{v}\cdot\frac{\partial\mu}{\partial\boldsymbol{x}} = 0\,, \tag{67}$$

which do not follow from the condition that the class of $\mathrm{d}\Omega$ is less than 4.

The non integrability of Ω is characterized by the changes of value which Ω suffers when its infinitesimal variation is integrated along closed paths. Let C be a closed path in space-time and C_0 the closed curve formed by the positions, at the time 0, of the elements of fluid lying on C. It follows from the well known Weber equations that

$$\boldsymbol{v}\cdot\delta\boldsymbol{x} - \boldsymbol{v}_0\cdot\delta\boldsymbol{x}_0 = \delta\int_0^t\left(\frac{\boldsymbol{v}^2}{2} - W - \int\frac{\mathrm{d}p}{\varrho}\right)\mathrm{d}t\,. \tag{68}$$

$\boldsymbol{v}_0$ and $\boldsymbol{x}_0$ denote the velocity and the position vector at the time 0 of the element of fluid whose velocity and position vector at the time t are $\boldsymbol{v}$ and $\boldsymbol{x}$, respectively. The δ denote isochronous variations. Since

$$\mathrm{d}\boldsymbol{x} = \delta\boldsymbol{x} + \boldsymbol{v}\,\mathrm{d}t\,, \qquad \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} = \boldsymbol{v}\cdot\delta\boldsymbol{x} + \boldsymbol{v}^2\,\mathrm{d}t\,, \tag{69}$$

we have

$$\mathrm{d}\Omega = \boldsymbol{v}\cdot\delta\boldsymbol{x} + \left(\frac{\boldsymbol{v}^2}{2} - W - \int\frac{\mathrm{d}p}{\varrho}\right)\mathrm{d}t\,, \tag{70}$$

$\delta\boldsymbol{x}$ denoting the displacement from the fluid element which is at the position $\boldsymbol{x}$ at the time t to the position at the time t of the fluid element which is at the position $\boldsymbol{x}+\mathrm{d}\boldsymbol{x}$ at the time $t+\mathrm{d}t$. It follows from (70) and (68) that

$$\mathrm{d}\Omega = \boldsymbol{v}_0\cdot\delta\boldsymbol{x}_0 + \mathrm{d}\int_0^t\left(\frac{\boldsymbol{v}^2}{2} - W - \int\frac{\mathrm{d}p}{\varrho}\right)\mathrm{d}t\,. \tag{71}$$

Hence

$$\int_C \mathrm{d}\Omega = \int_{C_0}\boldsymbol{v}_0\cdot\delta\boldsymbol{x}_0 = \int_{C_0}\delta\Omega_0\,. \tag{72}$$

Equation (72) shows that the variation of Ω along a closed path in space-time is the same for all curves C formed by the same fluid elements and does not depend on the instants in which the trajectories of the elements are crossed. Equation (72) expresses the Cartan generalization [6] of the theorem of the circulation of Lord Kelvin, which corresponds to the particular case of a path C of isochronous positions.

5. – Equations of the Helmholtz vortex theorems to the general theory.

The generalization of the gauge invariant field formalism discussed in section 1 corresponds to the introduction of the vorticity in hydrodynamics. In the particular case of the non relativistic Schrödinger equation for a spinless particle, the similarity with the hydrodynamical problem is so complete that the generalization can be performed by the consideration of the vortex motions of the Madelung fluid. In this particular case the extension of the Helmholtz theorems on the conservation of the vorticity is straightforward, as indicated in I. We shall now see that the introduction of the non integrable quantities

[6] E. Cartan: *Leçons sur les invariants intégraux* (Paris, 1922), p. 22.

allows us to extend the Helmholtz theorems to any gauge invariant field theory, the vorticity being replaced by the non-integrability vector $\boldsymbol{\eta}$

$$\boldsymbol{\eta} = \frac{\partial \lambda}{\partial \boldsymbol{x}} \wedge \frac{\partial \mu}{\partial \boldsymbol{x}} . \tag{73}$$

Thus the theorems on the conservation of the vorticity are seen to be associated with the gauge invariance.

The introduction of the Clebsch parameters λ and μ in the general theory of section 1 corresponds to the replacement of the integrable quantities $S^{(r)}$ by non integrable quantities $\Omega^{(r)}$ having definite difference of values at two neighbouring space-time points

$$\mathrm{d}\Omega^{(r)} = \mathrm{d}S^{(r)} + \lambda \, \mathrm{d}\mu . \tag{74}$$

Since

$$\mathrm{d}(\Omega^{(r)} - \Omega^{(s)}) = \mathrm{d}(S^{(r)} - S^{(s)}) = \text{exact differential}, \tag{75}$$

the differences $\Omega^{(r)} - \Omega^{(s)}$ are integrable quantities.

In the general formalism, we can define a velocity $\boldsymbol{v}$ of the charge at each point

$$j_l = v_l j_0 . \tag{76}$$

Once $\boldsymbol{v}$ is known, the trajectories of the elements of charge are defined by the differential equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}(t, \boldsymbol{x}) . \tag{77}$$

The equations (15) express simply that λ and μ are constants of the motion of the charge elements.

The solution of the equation (77) expresses the values of the coordinates of a charge element at the time t as functions of t and the position vector $\boldsymbol{x}_0$ of the charge element at some initial time, at the time 0 for instance

$$\boldsymbol{x} = \boldsymbol{\xi}(t, \boldsymbol{x}_0) . \tag{78}$$

λ and μ, being constants of the motion, have the same values at all the space-time points on a trajectory

$$\lambda(t, \boldsymbol{x}) = \lambda(0, \boldsymbol{x}_0) , \qquad \mu(t, \boldsymbol{x}) = \mu(0, \boldsymbol{x}_0) . \tag{79}$$

Hence

$$\int_c \lambda \, \mathrm{d}\mu = \int_{c_0} \lambda(0, \boldsymbol{x}_0) \, \mathrm{d}\mu(0, \boldsymbol{x}_0) , \tag{80}$$

C and C_0 being the closed curves of equation (72). It follows from (74) and (80) that there are equations of the type (72) in the general case

$$\int_C \mathrm{d}\Omega^{(r)} = \int_{C_0} \mathrm{d}\Omega^{(r)}(0, \boldsymbol{x}_0)\,. \tag{81}$$

Thus we get the extension of the circulation theorems of Cartan and Lord Kelvin.

Let C be formed by isochronous positions of the charge elements and Σ a surface limited by the contour C, the Stokes theorem shows that

$$\int_C \mathrm{d}\Omega^{(r)} = \int_C \lambda\,\mathrm{d}\mu = \int_\Sigma \boldsymbol{n}\cdot\boldsymbol{\eta}\,\mathrm{d}\Sigma\,, \tag{82}$$

$\boldsymbol{n}$ denoting the unit vector on the normal to Σ, with a convenient orientation.

Let us introduce the $\boldsymbol{\eta}$-lines, i.e. the lines of force of the $\boldsymbol{\eta}$-field. We shall call $\boldsymbol{\eta}$-tubes the tubes formed by $\boldsymbol{\eta}$-lines. The strength of a $\boldsymbol{\eta}$-tube can be defined as the flux of the vector $\boldsymbol{\eta}$ through any section of the tube, the flux being independent of the choice of the section, as a consequence of the equation $\operatorname{div}\boldsymbol{\eta} = 0$. A $\boldsymbol{\eta}$-surface is formed by $\boldsymbol{\eta}$-lines. The properties of the $\boldsymbol{\eta}$-lines and $\boldsymbol{\eta}$-surfaces are similar to those of the vortex lines and vortex surfaces of an inviscid barotropic fluid and can be established in the same way:

I) *The charge elements lying at one instant of time on a $\boldsymbol{\eta}$-line (or $\boldsymbol{\eta}$-surface) form also an $\boldsymbol{\eta}$-line (or $\boldsymbol{\eta}$-surface) at any other instant of time.*

II) *The strength of a $\boldsymbol{\eta}$-tube is time independent.*

III) *A necessary and sufficient condition for a surface to be a $\boldsymbol{\eta}$-surface is that $\int_C \mathrm{d}\Omega^{(r)}$ be null for any closed line C of the surface which can be reduced continuously to a point by a deformation on the surface.*

IV) *A necessary and sufficient condition for a surface to be a $\boldsymbol{\eta}$-surface is that $d\Omega^{(r)}$ be the exact differential of a function of two variables on the surface.*

It follows from (79) that

$$\lambda\,\mathrm{d}\mu = \lambda_0\,\mathrm{d}\mu_0\,, \qquad (\lambda_0 = \lambda(0, \boldsymbol{x}_0),\ \mu_0 = \mu(0, \boldsymbol{x}_0)). \tag{83}$$

We shall assume that d corresponds now to an isochronous displacement and introduce a second displacement δ of the same kind, commutable with d. It follows from (83) that

$$\delta\lambda\,\mathrm{d}\mu - \mathrm{d}\lambda\,\delta\mu = \delta\lambda_0\,\mathrm{d}\mu_0 - \mathrm{d}\lambda_0\,\delta\mu_0 \tag{84}$$

By taking into account that the displacements d and δ are isochronous, (84) becomes

$$\boldsymbol{\eta}\cdot\mathrm{d}\boldsymbol{x}\wedge\delta\boldsymbol{x} = \boldsymbol{\eta}_0\cdot\mathrm{d}\boldsymbol{x}_0\wedge\delta\boldsymbol{x}_0\,. \tag{85}$$

Let $\Delta \boldsymbol{x}$ be an infinitesimal vector non coplanar with $\mathrm{d}\boldsymbol{x}$ and $\delta \boldsymbol{x}$ and ϱ the charge density.

Since

$$\varrho\, \Delta \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} \wedge \delta \boldsymbol{x} = \varrho_0\, \Delta \boldsymbol{x}_0 \cdot \mathrm{d}\boldsymbol{x}_0 \wedge \delta \boldsymbol{x}_0 \,, \tag{86}$$

as a consequence of the conservation of the charge and $\Delta \boldsymbol{x}$ is expressed in terms of the corresponding initial displacement by the formula

$$\Delta \boldsymbol{x} = \mathcal{K}\, \Delta \boldsymbol{x}_z \qquad (\mathcal{K} = \text{linear operator})$$

we get from (85) and (86)

$$\frac{\boldsymbol{\eta}}{\varrho} = \mathcal{K}\, \frac{\boldsymbol{\eta}_0}{\varrho_0} \,, \tag{87}$$

because $\mathrm{d}\boldsymbol{x}$ and $\delta \boldsymbol{x}$ are arbitrary infinitesimal vectors. We have obviously

$$\Delta \boldsymbol{x} = \left(\Delta \boldsymbol{x}_0 \cdot \frac{\partial}{\partial \boldsymbol{x}_0} \right) \boldsymbol{x}(t, \boldsymbol{x}_0) \,, \tag{88}$$

for any $\Delta \boldsymbol{x}_0$. Therefore the form of $\mathcal{K}$ is determined and we get

$$\frac{\boldsymbol{\eta}}{\varrho} = \left(\frac{\boldsymbol{\eta}_0}{\varrho_0} \cdot \frac{\partial}{\partial \boldsymbol{x}_0} \right) \boldsymbol{x}(t, \boldsymbol{x}_0) \,. \tag{89}$$

Equation (89) gives the extension of the well known Cauchy theorem to the general theory: *The vector $\boldsymbol{\eta}/\varrho$ varies as a line element $\Delta \boldsymbol{x}$ of the charge distribution.*

The Cauchy formula shows that a line element in the charge distribution, which is parallel to $\boldsymbol{\eta}_0$ at the time 0, will be parallel to $\boldsymbol{\eta}$ at the time t, as it is carried along by the charge. Hence the charge elements lying on a $\boldsymbol{\eta}$-line at one instant of time lie also on a $\boldsymbol{\eta}$-line at any other instant. Since the $\boldsymbol{\eta}$-surfaces are formed by $\boldsymbol{\eta}$-lines, there is a similar property for these surfaces. Thus we have proven the theorem I. Equation (85) shows that the flux of $\boldsymbol{\eta}$ through the moving surface element defined by $\mathrm{d}\boldsymbol{x}$ and $\delta \boldsymbol{x}$ is time independent, so that the strength of an infinitesimal $\boldsymbol{\eta}$-tube does not vary with time. Any $\boldsymbol{\eta}$-tube can be divided into infinitesimal $\boldsymbol{\eta}$-tubes and we get the theorem II.

The theorem III is an immediate consequence of (82), because $\boldsymbol{\eta} \cdot \boldsymbol{\eta}$ must be 0 everywhere on the surface when $\int_C \mathrm{d}\Omega^{(r)} = 0$ for any curve C of the surface which can be reduced continuously to a point.

Let u_1 and u_2 be two parameters (curvilinear coordinates) on a surface Σ. On the surface Σ

$$\mathrm{d}\Omega^{(r)} = W_1^{(r)}\, \mathrm{d}u_1 + W_2^{(r)}\, \mathrm{d}u_2 \,. \tag{90}$$

In order that $\mathrm{d}\Omega^{(r)}$ be an exact differential on Σ, it is necessary and sufficient that

$$\frac{\partial}{\partial u_1} W_2^{(r)} - \frac{\partial}{\partial u_2} W_1^{(r)} = \boldsymbol{\eta} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u_1} \wedge \frac{\partial \boldsymbol{x}}{\partial u_2} \right) = 0 \,, \tag{91}$$

i.e. that $\boldsymbol{\eta}$ should be orthogonal to the normal everywhere on Σ. Thus we get the theorem IV.

6. – Non-integrable phases and electromagnetic forces.

In the cases in which L can be expressed in terms of the complex quantities $U^{(r)}$ and $U^{*(r)}$, the method of generalization of section **1** corresponds to the introduction of non integrable phases. The possibility of introducing non integrable phases in the quantum mechanics was already discussed by DIRAC (7) a long time ago. DIRAC showed that the interaction with the electromagnetic field can be taken into account by using non integrable phases, the procedure allowing also to treat the motion of charged particles in the field of magnetic monopoles.

The use of the Clebsch parameters appears to be advantageous in the discussion of the interaction with the electromagnetic field. It is always possible to express the potentials A_l in terms of two pairs of Clebsch parameters λ_s, μ_s

$$\frac{\varepsilon'}{c} A_l = - \sum_{s=1}^{2} \lambda_s \frac{\partial \mu_s}{\partial x_s} \,, \tag{92}$$

since this amounts to put the Pfaff expression $\sum_{l=0}^{3} A_l \, \mathrm{d}x_l$ in a canonical form

$$\frac{\varepsilon'}{c} \sum_{l=0}^{3} A_l \, \mathrm{d}x_l = - \sum_{s=1}^{2} \lambda_s \, \mathrm{d}\mu_s \,. \tag{93}$$

The Clebsch parameters λ_s, μ_s give a new description of the electromagnetic field.

The description of the electromagnetic field by means of the parameters λ_s, μ_s seems to be of particular convenience for the discussion of the gauge invariance. Let us consider the contact transformation

$$\sum_{s=1}^{2} \lambda_s \, \mathrm{d}\mu_s - \sum_{s=1}^{2} \lambda'_s \, \mathrm{d}\mu'_s = \mathrm{d}\chi \,. \tag{94}$$

The contact transformation (94) gives rise to a gauge transformation of the

(7) P. A. M. DIRAC: *Proc. Roy. Soc.*, A 133, 60 (1931).

potentials

$$A_l' = A_l + \frac{c}{\varepsilon'}\frac{\partial \chi}{\partial x_l}\,. \tag{95}$$

The gauge invariance of the second kind of the electromagnetic field is associated to the group of the contact transformations of the Clebsch parameters λ_s, μ_s.

It is interesting to notice that the sub-group of the homogeneous contact transformations of the λ_s, μ_s

$$\sum_{s=1}^{2}\lambda_s \mathrm{d}\mu_s = \sum_{s=1}^{2}\lambda_s' \mathrm{d}\mu_s'\,, \tag{96}$$

leaves the potentials A_l invariant. *This shows that the group of the contact transformations of the* λ_s, μ_s *is richer than that of the gauge transformation of the potentials* A_l.

Let us consider now the simplest case in which L involves only a single phase S. The most general non integrable phase Ω can be obtained by means of two pairs of Clebsch parameters, but we may also write

$$\mathrm{d}\Omega = \mathrm{d}S + \sum_{s=1}^{2}\lambda_s \mathrm{d}\mu_s\,. \tag{97}$$

When the λ_s, μ_s are all independent, S must be a function of them, since the class of $\mathrm{d}\Omega$ cannot be higher than 4. By introducing potentials A_l defined by a formula of the type (92), we get

$$\mathrm{d}\Omega = \mathrm{d}S - \frac{\varepsilon'}{c}\sum_{l=0}^{3}A_l \mathrm{d}x_l\,. \tag{98}$$

Thus we get Dirac's result on the possibility of describing the interaction with the electromagnetic field by using non integrable phases, since (98) shows that to replace S by the non integrable quantity Ω amounts to replace the S_l by the $S_l - (\varepsilon'/c)A_l$.

In the general case of several phases $S^{(r)}$, the differences between the Ω must be integrable, so that

$$\mathrm{d}\Omega^{(r)} = \mathrm{d}\Omega^{(1)} + \mathrm{d}\sigma^{(r)} \qquad (r \neq 1) \tag{99}$$

the $\mathrm{d}\sigma^{(r)}$ being exact differentials. The Pfaff form $\mathrm{d}\Omega^{(1)}$ can be put in the quasi-canonical form

$$\mathrm{d}\Omega^{(1)} = \mathrm{d}S^{(1)} + \sum_{s=1}^{2}\lambda_s \mathrm{d}\mu_s = \mathrm{d}S^{(1)} - \frac{\varepsilon'}{c}\sum_{l=0}^{3}A_l \mathrm{d}x_l \tag{100}$$

and by introducing the $S^{(r)}$

$$\mathrm{d}\sigma^{(r)} = \mathrm{d}S^{(r)} - \mathrm{d}S^{(1)} \qquad (r \neq 1) \tag{101}$$

we get for all the $\mathrm{d}\Omega^{(r)}$

$$\mathrm{d}\Omega^{(r)} = \mathrm{d}S^{(r)} - \frac{\varepsilon'}{c}\sum_{l=0}^{3} A_l \mathrm{d}x_l \,. \tag{102}$$

Equation (102) shows that the interaction with the electromagnetic field can be described by means of non integrable phases, whose differences are integrable. The gauge invariance condition (25) means precisely that L depends on the various phases only through their differences, no restrictions being imposed on the dependence on the $\partial S^{(r)}/\partial x_l$.

The above considerations refer to the non generalized U field interacting with the electromagnetic field. In the case of the generalized U field, the non integrability of the phases is not entirely due to the interaction with the electromagnetic field.

RIASSUNTO (*)

Il metodo di generalizzazione applicato in I alle equazioni d'onda di Schrödinger e Dirac e all'equazione classica di Hamilton-Jacobi si estende a tutte le teorie derivate da principi variazionali che ammettono la « gauge invariance » di prima specie. Con una lieve modificazione della definizione ordinaria della trasformazione di gauge di prima specie, vi si includono anche la teoria di Hamilton-Jacobi e la dinamica dei liquidi barotropici non viscosi. In tutti i casi il metodo di generalizzazione corrisponde al passaggio dai moti irrotazionali ai moti vorticosi dei fluidi non viscosi, benchè il ruolo della vorticità sia assunto da altre grandezze. Si dimostra che si possono estendere alla generalizzazione del formalismo del campo gauge-invariante teoremi analoghi a quelli di Helmholtz, Kelvin e Cartan e la formola di Cauchy della teoria del moto vorticoso di un fluido barotropico non viscoso, la vorticità essendo sostituita da un vettore di non integrabilità che compare nel formalismo. Il metodo di generalizzazione può considerarsi consistere nella sostituzione di grandezze simili alle fasi delle funzioni d'onda per mezzo di grandezze non integrabili.

(*) *Traduzione a cura della Redazione.*

Life Time Measurements of Unstable Charged Particles of Cosmic Radiation Using Emulsions.

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(ricevuto il 10 Agosto 1954)

Summary. — In the present paper two different procedures are described which both allow the determination of the mean life of charged unstable particles whose tracks are observed in nuclear emulsions. The first procedure, which has been already presented in a preceding paper, is based on the method of « maximum likelihood ». The second procedure is based on the use of the so called « residual time ». This time is defined as follows: Let us consider a particle which decays in flight in the plate, with a velocity β_d at an instant t_d. If it has not decayed it would have been brought to rest in the emulsion at a time t_0. The time interval $t_d - t_0$ is the « residual time » (see fig. 1). The combination of the two methods reduces the influence of experimental bias of both geometrical and kinematical origin. The application of these methods to the case of hyperons allows one to establish the following limits to their mean life:

$$3 \cdot 10^{-10} \geqslant \tau \geqslant 0.5 \cdot 10^{-10} \text{ s}.$$

1. – Introduction.

The measurement of the life time of unstable charged particles of cosmic radiation using stripped nuclear emulsion is not, in general, a simple problem mainly for the following reasons. Cosmic rays are an uncontrolled source of radiation which produces a large variety of particles having different mass, charge and energy. Many of these particles behave in similar ways so that their identification is possible only through accurate measurements on the track which they produce in the emulsion. It is therefore necessary to consider only those events for which the length of the track is sufficient to determine unambiguously the mass, charge and energy of the particle. This mi-

nimum length is ~ 1 mm in the most favorable cases and fixes a lower limit for the life of the particle which has to survive for the time necessary to cover this distance. For particles having a kinetic energy of 1 GeV and a mass comparable with that of a proton the lower limit is $\sim 10^{-12}$ s.

On the other hand, an upper limit is fixed by the time which the particles spend in the emulsion before being brought to rest. If their life time is appreciably longer than this, the majority of the particles will be brought to rest before they decay: very few decays in flight will be observed and no estimate of their mean life will be possible in practice. For the same type of particles as considered before, the upper limit for « measurable life times » is $\sim 10^{-9}$ s.

The use of nuclear emulsion as a detector, which has many advantages with respect to all the other instruments used in this type of research, is however likely to introduce some experimental bias which might affect considerably the result of a measurement of the life time. In fact, the detection of a decaying particle is much more difficult if the particle decays when in flight rather than at rest. As a consequence a measured mean life might appear longer than it is in reality. An estimate of the relative efficiency of detection of the different types of events is necessary if an accurate measurement is required.

In the present paper we intend to discuss a method which has been used previously and a new one which has been elaborated recently in Rome.

2. – Maximum likelihood method (ML-Method).

This method has been already described in a paper presented at the Padua conference [1]: it is essentially a modification of a method discussed by M. S. Bartlett [2] in connection with cloud chamber experiments. We shall summarize here its basic ideas.

Let us consider a number of observations all of one type of particle, decaying in the volume of a stack of emulsion. Let l_r be the length of the r-th track observed in the emulsion from the point of creation of the particle (or of its entry in the emulsion) to the point of decay. Let us denote by t_r the time spent by the r-th particle to cover l_r and by T_r the total time available for the same particle for decay within the emulsion.

If τ is the life time of the particles considered and we have a number n of observations, for each of which the values of t_r and T_r are known, the

[1] C. Castagnoli, G. Cortini and C. Franzinetti: *Report at the Padua Conference* (in press).

[2] M. S. Bartlett: *Phil. Mag.*, **44**, 249 (1953).

most probable value τ_m of τ is that which (see reference (1)) is given by the equation

$$(1) \qquad \tau_m = \frac{1}{n_b + n_c}\left\{\sum_1^{n_a} t_i + \sum_1^{n_b} t_k + \sum_1^{n_c}\left(t_l + \frac{T_l}{\exp\left[T_l/\tau_m\right]-1}\right)\right\},$$

where n_a represents the number of particles which are brought to rest in the emulsion before decaying; n_b those which decay in flight when they have such a small velocity that they would have been brought to rest in the emulsion if they had not decayed; n_c those which decay in flight in the emulsion and are not to be included amongst the n_b .

As mentioned in the previous paragraph each event is associated with a different probability of being detected, which depends on the ionization density at the point of decay.

Equation (1) could then be written as follows

$$(2) \qquad \tau_m = \frac{1}{\sum_1^{n_b} 1/p_k + \sum_1^{n_c} 1/p_l}\left\{\sum_1^{n_a} t_i + \sum_1^{n_b} t_k/p_k + \sum_1^{n_c}\left(t_l + \frac{T_l}{\exp\left[T_l/\tau_m\right]-1}\right)\frac{1}{p_l}\right\},$$

the p's being the relative probability for each event of being observed. The evaluation of the p's is in general very difficult. If the p's are put equal to 1 the value of τ_m which one gets from (1) is likely to be higher than the real one.

This method does not require in general the knowledge of the energy spectrum of the particles.

However one can easily show that, if the source of particles is constant with respect to the variable t (that means an energy spectrum at the origin of the type $\sim T\,\mathrm{d}T$) the kinematic bias does not affect the value of the mean life time obtained with the ML-method.

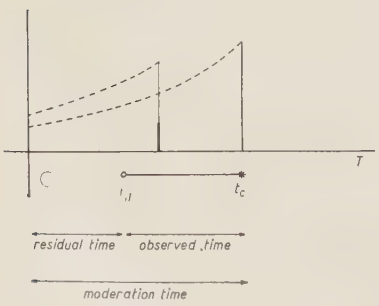

Fig. 1.

3. – Residual time method (RT-method).

Let us consider, at first, a monocromatic source of particles. Let β be their velocity, R their residual range in the emulsion and t the time spent before they are brought to rest (moderation time) (see figure 1). If they are unstable and τ is their mean life time, the number of those which decay at rest is

$$N_0 = N \exp\left[-t/\tau\right]$$

and

$$N' = N(1 - \exp[-t/\tau])$$

that of those which decay in flight. Let us subdivide t in k equal intervals and let us direct the time axis in such a way that $t = 0$ corresponds to the instant in which the particles would be brought to rest if they had not decayed in flight. Calling t_c the time of creation of a particle and t_d the time of decay (see fig. 1), the number of particles which decay in the k-th and in the $(k - r)$-th interval are respectively

$$N_k = N(1 - \exp[-\theta/\tau])$$

and

$$N_{k-r} = N \exp[-r\theta/\tau] \cdot (1 - \exp[-\theta/\tau])$$

where $\theta = t_c/K$. It follows that

(3) $$N_{k-r} = N_k \exp[-r\theta/\tau] \qquad r = 1, 2, \ldots k-1$$

and

(4) $$N_{l+1}/N_l = \exp[\theta/\tau] \qquad l = 1, 2, \ldots k-1 .$$

Also

(5) $$\frac{N_1}{N_0} = \frac{\exp[-(k-1)\theta/\tau]}{\exp[-k\theta/\tau]} (1 - \exp[-\theta/\tau]) = \exp[\theta/\tau] - 1$$

the N_l's, introduced in these formulae, are the numbers of the existing particles which decay in the interval θ_l. If some of them are lost in the observation of the emulsion and n_r are the observed numbers, the above formulae read

(3′) $$\frac{n_{k-r}}{n_k} = \frac{\eta_{k-r}}{\eta_k} \exp[-r\theta/\tau]$$

(4′) $$\frac{n_{l+1}}{n_l} = \frac{\eta_{l+1}}{\eta_l} \exp[\theta/\tau]$$

(5′) $$\frac{n_1}{n_0} = \frac{\eta_1}{\eta_0} (\exp[\theta/\tau] - 1)$$

where the $\eta_l = n_l/N_l$ are the « efficiency of detection » associated with particles of velocity and ionizing power corresponding to the interval θ_l .

An estimate of the η's can be done and it will be done later. In a first approximation we can, however, consider $\eta_0 = \eta_1 = 1$ and calculate an order of magnitude for τ. This assumption is certainly not far wrong if θ is taken sufficiently small. We get then an approximate value of

$$\frac{1}{\tau_0} = \frac{1}{\theta} \ln \left(1 + \frac{N_1}{N_0}\right).$$

Using (4′) and the value of τ just obtained, an estimate can be made of the η_i's. We get

$$\eta_2 = \frac{n_2}{n_1(1 + n_1/n_0)} \qquad \eta_3 = \frac{n_3}{n_1(1 + n_1/n_0)^2} \ldots \text{etc.}$$

The values of the η's so calculated can be used in connection with method I.

We would like to point out however that the introduction of the η's in the RT-method is simpler than that of the p's in the other. In this case the time scale origin coincides with the origin of the β-axis. Therefore each time interval θ can be associated with a value of the ionization density and so of η. In formula (1′) or (2′) this was not possible as the origins of the time measurements were associated with a different value of β each time and the p's had to be calculated for each individual event.

4. – Effect of a non-monocromatic source on the measurements.

So far we have considered a monocromatic source of particles all created with a velocity β and therefore associated with the same time $t = k\theta$. In cosmic ray research however this case seldom happens. We can, of course, restrict ourselves to consider only those particles which were created at a time $t_c > k\theta$ (in the time scale established in fig. 1): the situation is then analogous to that of a monocromatic source. In this way, however, one might be forced to reduce appreciably the number of events and as a consequence is that the statistical error of the final result is much larger.

This can be avoided if the velocity spectrum of the particles which are considered is known. Let $dr = G(\beta)\,\mathrm{d}\beta$ be the number of particles created with a velocity β. Considering β as a function of the slowing down time in emulsion, $\mathrm{d}r = G(\beta)\,\mathrm{d}\beta = S(t)\,\mathrm{d}t$, the number of particles which decay between t and $t + \mathrm{d}t$ is

$$\mathrm{d}n\,(t) = \frac{\mathrm{d}t}{t} \exp\,[t/\tau] \left\{ \mathcal{N} \exp\,[-k\theta/\tau] + \int_t^{k\theta} \exp\,[-x/\tau]\, S(x)\,\mathrm{d}x \right\},$$

were $\mathcal{N}$ is the number of particles created at a time $t > k\theta$.

It follows

$$N_k = \int_{t=(k-1)\theta}^{t=k\theta} \mathrm{d}n \,,$$

which leads to equations equivalent to (3), (4) and (5). If we consider the simple case of a source the intensity of which is constant over the time interval $0 \to \theta$, we obtain for the number of particles decaying at rest and in the interval $0 \to \theta$

$$N_0 = S\tau + (\overline{N} - S\tau) \exp [-\theta/\tau] \,, \tag{9}$$

$$N_1 = S\theta + (\overline{N} - S\tau)(1 - \exp [-\theta/\tau]) \,, \tag{10}$$

where S is the number of particles created per unit time between $t = 0$ and $t = \theta$ and $\overline{N}$ is the number of particles existing at the time $t = \theta$.

5. – Independent estimate of the efficiencies.

In a tentative way one can assume that the probability of observing a particle decaying at a time t (the time being always referred to the scale of fig. 1) is a known function $p(t)$. From the definition of the η's one gets obviously (for a monochromatic source)

$$\eta_l = \frac{\int_{(l-1)\theta}^{l\theta} \exp [t/\tau]\, p(t)\, \mathrm{d}t}{\int_{(l-1)\theta}^{l\theta} \exp [t/\tau]\, \mathrm{d}t} \,, \tag{11}$$

$p(t)$ is expected to be a decreasing function of t. For $t = 0$ one can put $p(0) = 1$. For sake of simplicity we assume that $p(t) \sim \exp [-t/T]$ were T is a parameter which we shall determine later. Then

$$\eta_l = \frac{T}{T+\tau} \frac{\exp \left[((1/\tau) - (1/T))\,\theta\right] - 1}{\exp [\theta/\tau] - 1} \exp [-(l-1)\theta/T] \tag{12}$$

$$\eta_{l+1}/\eta_l = \exp [-\theta/T] \qquad l = 1, 2, \ldots, k-1 \,.$$

T can be adjusted to fit the values of η_3/η_2 and η_2/η_1 measured as described previously. Once this is done the individual values of the η's can be calculated using equation (12). These can be introduced into equation (4′) or (5′) and a more accurate value of τ is obtained. Successive approximations can be

calculated if desired. We want to emphasize that all one can get out of these considerations is an order of magnitude. But criticizable as might appear this attempt, as long as we shall deal with rare events, such as hyperons a rough estimate of the efficiency in the way which we have outlined is not unreasonable.

6. – Comparison between the two methods.

The main features of the two methods can be summarized as follows.

The ML-method. – It requires the knowledge of the observed time t_r and of the so called potential time T_r. t_r can be evaluated knowing the velocity of the particle; T_r is calculated knowing the velocity β_d at decay and the geometry of the event in the stack of emulsions. No knowledge of the energy spectrum of the particles concerned is necessary; on the other hand it is very difficult to take any account of the experimental bias deriving from the different efficiency of detection associated with events of different velocity. This introduces a systematic error which makes the value of the measured life time greater than it is. Therefore if no correction for systematic losses is made the value of τ obtained is to be considered rather an upper limit.

The RT-method. – It requires the knowledge of the « moderation time », as defined before and that of the residual time. The knowledge of the energy spectrum at production of the particles considered is not essential as long as one restricts the measurements to those particles which are produced with sufficiently high velocity. An estimate of the systematic loss is possible: once the relative efficiencies η are known they can be used to calculate τ with ML-method. We would like to mention another point. A graph as that sketched in fig. 1 has a meaning as long as the ranges associated with the moderation time of the particles considered are small compared with the size of the emulsion stack. If this condition is not verified a geometrical bias is introduced which favours the particles created at low energy which have a higher chance of decaying in the emulsion.

7. – Application of these methods to measure the life time of hyperons.

The ML-method has already been applied to measure the lifetime of hyperons ([1]) and the result was

$$\tau = 2.9^{+4.8}_{-1.1} \cdot 10^{-10} \text{ s} ,$$

when no account of any systematic loss was taken.

Using the RT-method, with the data indicated in fig. 2, (taking into account the losses due to different probability of observation and the effect of the source as outlined in the previous paragraphs) we get

$$0.5 \cdot 10^{-10} \leqslant \tau \leqslant 3 \cdot 10^{-10} \text{ s} .$$

The geometrical bias discussed in the previous paragraph is, in this case, negligible.

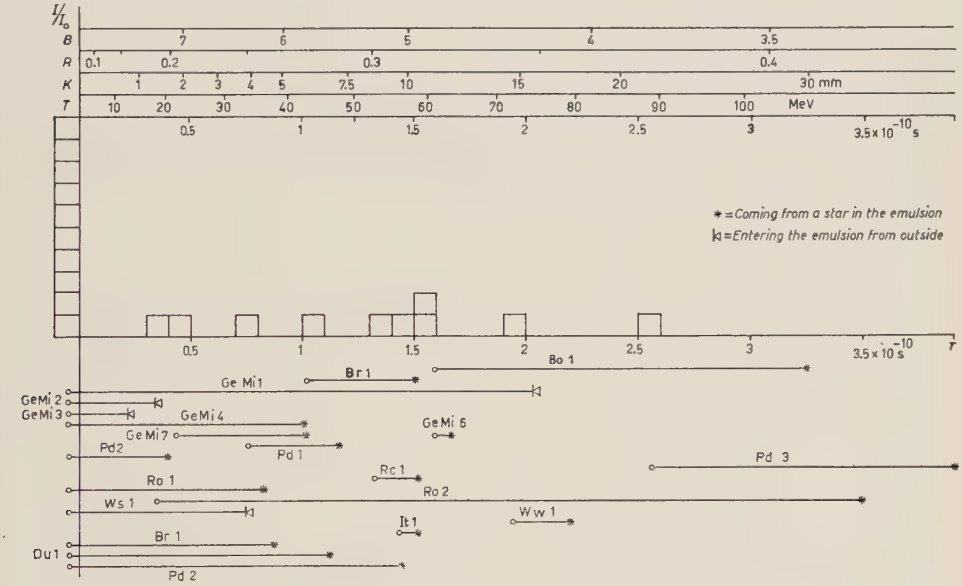

Fig. 2.

8. – Conclusion.

As one sees, the results obtained by different methods and making different assumptions, are consistent with a unique value which is most probably between

$$3 \cdot 10^{-10} \geqslant \tau \geqslant 0.5 \cdot 10^{-10} .$$

A value of τ below $0.5 \cdot 10^{-10}$ seems to us very improbable even taking the most pessimistic view about the systematic loss of events ($\eta_1 = 0.19$) and the influence of the energy spectrum.

One sees that the range of the probable values of the life time justifies the use of stripped emulsions for the measurement of the life time of hyperons.

RIASSUNTO

Il presente lavoro è dedicato alla esposizione e al confronto di due procedimenti per la determinazione della vita media di corpuscoli carichi instabili le cui tracce vengono osservate in emulsioni nucleari. Il primo procedimento, che già è stato esposto in un precedente lavoro, è basato sul metodo della « massima veorimiglianza » (*maximum likelihood*). Il secondo procedimento, invece, è basato sul far uso, come variabile per rappresentare il comportamento dei corpuscoli dall'istante in cui escono da una stella o entrano nella emulsione fino all'istante in cui decadono, del cosiddetto « tempo residuo », ossia del tempo che essi impiegherebbero, se non decadessero, a ridursi in quiete a partire dall'istante in cui decadono. Dopo aver confrontato tra loro i due procedimenti e aver mostrato che, specialmente se combinati insieme, essi permettono di eliminare sia l'influenza delle dimensioni finite del pacco di emulsioni che l'effetto della limitata efficienza di osservazione dipendente dalle condizioni cinematiche in cui avviene il decadimento, essi vengono applicati al caso degli iperoni. Viene così dimostrato che la corrispondente vita media è verosimilmente compresa tra i seguenti limiti: $3 \cdot 10^{-10} \geqslant \tau \geqslant 0.5 \cdot 10^{-10}$ s.

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On Time Reversal.

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Summary. — It is pointed out that a satisfactory definition of time reversibility in quantum theory has not yet been given, so that for example the reasons for the exclusion of a system as irreversible are somewhat obscure. In this paper a definition of reversibility is given, which takes account of the fact that a system may be described by a whole set of equivalent Hamiltonians. Such a definition therefore excludes irreversible systems and not one Hamiltonian which might still be one of a set of Hamiltonians some of which are reversible according to the old set of prescriptions. Our definition is suggested by the correspondence principle and based entirely on the transformation properties of observable quantities. It is found that the new formulation is sufficient to exclude systems whose Hamiltonians had hithertho been labelled as irreversible — though the method of demonstration — owing to the generality of the postulates — is somewhat more complicated than the usual one.

Introduction.

The question of time reversal in quantised theories was first treated by Wigner (1). Later various authors used this concept for the treatment of two types of problem:

1) the reduction of the number of independent elements of the scattering matrix — reciprocity theorem — and 2) the exclusion of certain forms of

(1) E. P. Wigner: *Göttinger Nachr.*, 546 (1932).

interaction between fields or particles on account of their non-invariance under time reflection. An example for the treatment of the first type of problem we find in the work of COESTER ([2]) and of WATSON ([3]), who recently established relations between the amplitudes for photo meson production and meson-nucleon scattering. The second topic has been applied in the theory of β decay (GAMBA, CAIANIELLO, MICHEL and others ([4])) and is reviewed in two papers by LÜDERS ([5]).

On re-examining the whole question we have been impressed by the fact, that there does not exist a sufficiently general definition of time reversal. It is therefore the purpose of this paper to give such a definition for a general quantum mechanical system. This definition will be based on the correspondence principle. As regards the reciprocity theorem 1) it will be shown that the latter is not a direct consequence of reversibility. Such a theorem can be deduced even for a system which is not reversible in the sense of our definition; in practice however the determination of relations between the elements of the S-matrix is greatly simplified in the case of reversible systems.

The discussion which we present here will have three parts. The first will deal with reversibility in classical mechanics.

Particular attention is devoted to the Hamiltonian formulation of classical mechanics as the one which can more easily be translated into the quantum mechanical language. An example is discussed which clearly shows how cautious one has to be in discussing the time reversibility of a system by inspecting its Hamiltonian, due to the fact that there exists a whole class of Hamiltonians giving rise to the same equations of the motion. Finally a definition of time reversibility is given by introducing the notion of anticanonical transformations a concept which is readily translated into quantum-mechanical language.

The second part contains a quantum mechanical formulation of the problem. In section 2·1 a matrix K is introduced which, when operating on the wave function of a system satisfying a Schrödinger equation transforms it into a wave function satisfying a time reversed Schrödinger equation. This matrix K can be introduced and explicitely constructed for any system, whatever its (time independent) Hamiltonian is. In section 2·2 we show how, by making use of the matrix K we may define corresponding to any operator a reversed operator satisfying, in the Heisenberg description, reversed equations of motion. As we have said the matrix K exists for any (conservative) system,

([2]) F. COESTER: *Phys. Rev.*, **84**, 1259 (1951).

([3]) K. M. WATSON.: *Phys. Rev.*, **95**, 228 (1954).

([4]) A. GAMBA: *Nuovo Cimento*, **7**, 919 (1950); E. R. CAIANIELLO: *Nuovo Cimento*, **9**. 4 (1952) and previous papers; L. MICHEL: *Thèse*, 1953.

([5]) G. LÜDERS: *Zeits. f. Phys.*, **133**, 325 (1952); *Det. Kong. Dan. Vidensk. Selsk.*, **28**. 5 (1954).

so that its existence cannot be taken as a basis for a descrimination between time-reversible and time irreversible systems if we do not subject the matrix K to further conditions. In section 2·5 such conditions are established, on the basis of the correspondence principle for a non relativistic system of particles, thus leading to the possibility of discrimination between time reversible and time irreversible systems. But, independently from such a discrimination, the general properties of the matrix K can be used (section 2·4) in proving the reciprocity theorem or in other words in establishing relations between elements of the matrix S. Such relations are of a particularly simple form in the case of time reversible systems but exist also for systems which are not time reversible.

In part **3** the conditions to be satisfied by the matrix K in the case of a time reversible *field* are established always by the aid of the correspondence principle. In this discussion particular emphasis is given to the occupation numbers representation as the one in which the correspondence between classical and quantum mechanical properties may be most easily exhibited. The cases of a spinless boson field, of the electromagnetic field and of the Dirac field are discussed. It is then shown how the conditions on the matrix K are not satisfied in some cases e.g. in the case of bosons with simultaneous scalar and vector coupling to fermions, which thus constitutes an example of a time irreversible system. Some critical remarks are contained in the last section.

1. – Classical Theory.

1·1. *Reversibility of a System of Point-Particles.* – The equations of motion of a charged particle in an electrostatic external field or of a massive particle in a gravitational field possess a property of symmetry which is called invariance under time reversal. This property may be formulated in several equivalent ways and we shall in view of the extension to quantum theory choose the following: let $\boldsymbol{r}$ be the position vector of a particle and $\dot{\boldsymbol{r}}$ its velocity. If the equations of motion lead in a time interval T from a state $\boldsymbol{r}_1, \dot{\boldsymbol{r}}_1$ to a state $\boldsymbol{r}_2, \dot{\boldsymbol{r}}_2$, then in the same time interval they also lead from, $\boldsymbol{r}_2, -\dot{\boldsymbol{r}}_2$ to $\boldsymbol{r}_1, -\dot{\boldsymbol{r}}_1$. This property is a consequence of the fact that the equations of motion are invariant under the substitution $t \to -t$.

In view of the extension to quantum theory we want to express this property by using the Hamiltonian formalism. It is then easy to see that there always exists a Hamiltonian which describes the motion of the systems defined above, and which if $\boldsymbol{r}$ is the position vector of a particle and $\boldsymbol{p}$ is a set of canonically conjugate momenta is invariant under the substitution

$$\boldsymbol{r} \to \boldsymbol{r}\,, \qquad \boldsymbol{p} \to -\boldsymbol{p}\,. \tag{1}$$

The same is true for a system of particles interacting among themselves through electrostatic or Newtonian forces.

Let us next consider the case of a system of particles coupled by an electromagnetic field, which is considered as part of the system. The symmetry under time reversal may then be generalized by saying that there exists a Hamiltonian which is invariant under the substitutions:

$$\text{(2)} \qquad \begin{cases} \boldsymbol{r}^{(i)} \to \boldsymbol{r}^{(i)} & \boldsymbol{E}_{\mathrm{tr}}(\boldsymbol{r}) \to \boldsymbol{E}_{\mathrm{tr}}(\boldsymbol{r}) \\ \boldsymbol{p}^{(i)} \to -\boldsymbol{p}^{(i)} & \boldsymbol{H}_{\mathrm{tr}}(\boldsymbol{r}) \to -\boldsymbol{H}_{\mathrm{tr}}(\boldsymbol{r}) . \end{cases}$$

$\boldsymbol{r}^{(i)}$ is the position vector of the i-th particle, $\boldsymbol{E}_{\mathrm{tr}}$ and $\boldsymbol{H}_{\mathrm{tr}}$ are the transverse parts of the electric and magnetic fields respectively, the longitudinal field being eliminated by the usual canonical transformation. The invariance property (2) leads to the following generalisation of the definition of reversibility. If the equations of motion lead in a time T from a state $\boldsymbol{r}_1^{(i)}$, $\dot{\boldsymbol{r}}_1^{(i)}$, $\boldsymbol{E}_{\mathrm{tr}}^{(1)}(\boldsymbol{r})$, $\boldsymbol{H}_{\mathrm{tr}}^{(1)}(\boldsymbol{r})$ to a state $\boldsymbol{r}_2^{(i)}$, $\dot{\boldsymbol{r}}_2^{(i)}$, $\boldsymbol{E}_{\mathrm{tr}}^{(2)}(\boldsymbol{r})$, $\boldsymbol{H}_{\mathrm{tr}}^{(2)}(\boldsymbol{r})$, they will also lead from the state $\boldsymbol{r}_2^{(i)}$, $-\dot{\boldsymbol{r}}_2^{(i)}$, $\boldsymbol{E}_{\mathrm{tr}}^{(2)}(\boldsymbol{r})$, $-\boldsymbol{H}_{\mathrm{tr}}^{(2)}(\boldsymbol{r})$ to the state $\boldsymbol{r}_1^{(i)}$, $-\dot{\boldsymbol{r}}_1^{(i)}$, $\boldsymbol{E}_{\mathrm{tr}}^{(1)}(\boldsymbol{r})$, $-\boldsymbol{H}_{\mathrm{tr}}^{(1)}(\boldsymbol{r})$.

The symmetry property discussed in this section may be summarized briefly by introducing the notion of an anticanonical transformation, which is defined to be a transformation in which the Poisson brackets change sign. The property of reversibility considered here may then be expressed as the invariance of the Hamiltonian under the anticanonical transformation (1) or (2). It should be noted however, and we want to discuss this point in the next section, that (1) and (2) only represent special cases of anticanonical transformations.

1·2. *Reversibility and the Hamiltonian Formalism.* – In the previous section we have discussed the reversibility properties of a very special class of classical systems. In this section we shall try to give a definition of reversibility which should enable us to decide whether or not a given general classical system is reversible.

To illustrate the type of problem we are faced with, we shall first discuss an example. Consider a system of two particles interacting through a hypothetical interaction term $(\boldsymbol{pr})V(\boldsymbol{r})$, where $\boldsymbol{p}$ and $\boldsymbol{r}$ are the relative momenta and coordinates, $V(\boldsymbol{r})$ is a function of $\boldsymbol{r}$. The Hamiltonian of this system is

$$\text{(3)} \qquad H = \tfrac{1}{2}p^2 + (\boldsymbol{p}\cdot\boldsymbol{r})V(\boldsymbol{r})$$

and obviously this Hamiltonian is not invariant under the substitution (1). However, the equations of motion for the coordinates $\boldsymbol{r}$, which follow from the Hamiltonian (3) are invariant if t is replaced by $-t$, provided that

$$\text{(4)} \qquad \boldsymbol{r} \wedge \operatorname{grad} V = 0 .$$

We see therefore that it is not legitimate to conclude that a system is irreversible on the strength of the fact that one of the Hamiltonians describing it is not invariant under the substitution (1). Indeed there is an infinite number of Hamiltonians which lead to the same equations of motion. This infinity is due to the fact that even if we restrict one set of canonical variables to be the cartesian coordinates of the particles, there is still an infinite number of ways of choosing the canonically conjugated momenta $\boldsymbol{p}$. All the possible choices of $\boldsymbol{p}$ belong to the family

$$\boldsymbol{p}^{(i)\prime} = \boldsymbol{p}^{(i)} + \frac{\partial F}{\partial \boldsymbol{r}^{(i)}} \tag{5}$$

where F is an arbitrary function of the coordinates. If we make this substitution on the Hamiltonian (3), it is easy to see that with the choice

$$\text{grad } F = -2\boldsymbol{r}V\,, \tag{6}$$

(which is integrable provided that condition (4) is satisfied), the new Hamiltonian becomes

$$H' = \tfrac{1}{2}p'^2 - \tfrac{1}{2}r^2 V, \tag{3'}$$

which is invariant under the substitution (1) and gives of course the same equations of motion for $\boldsymbol{r}$ as the Hamiltonian (3).

In order to decide, therefore, whether or not a system is reversible one has either to check the equations of motion for the position coordinates, or to see whether the Hamiltonian is invariant under at least one of the transformations

$$\begin{cases} \boldsymbol{r}^{(i)} \to \boldsymbol{r}^{(i)} \\ \boldsymbol{p}^{(i)} \to -\boldsymbol{p}^{(i)} + \dfrac{\partial F}{\partial \boldsymbol{r}^{(i)}}\,. \end{cases} \tag{7}$$

with F an arbitrary function of the coordinates. Equation (7) represents the definition of reversibility in the Hamiltonian formalism which we shall adopt: a system is reversible, if its Hamiltonian is invariant under one of the anticanonical transformations (7). Equation (7) represents the most general anticanonical transformation which leaves the position coordinates invariant. One may also convince oneself quite easily that in the case of reversible non degenerate systems there exists only one such substitution (6).

(6) There always exists for every given conservative system an anticanonical transformation which leaves H invariant, so that, without the condition $\boldsymbol{r}^{(i)} \to \boldsymbol{r}^{(i)}$ no distinction between reversible and irreversible systems could be made. One anticanonical transformation is obtained by introducing action variables $I^{(i)}$ and angle variables $w^{(i)}$ and making the substitution:

$$I^{(i)} \to I^{(i)}\,, \qquad w^{(i)} \to -w^{(i)}\,.$$

Finally, we want to discuss the classical example of an irreversible system — a point particle in a magnetic field. Here the Hamiltonian is:

$$H = \tfrac{1}{2}(\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{r}))^2$$

where $\boldsymbol{A}$ is the vector potential of the magnetic field: curl $\boldsymbol{A} = \boldsymbol{H}$. With the substitution (7) H becomes

$$\hat{H} = \tfrac{1}{2}(-\boldsymbol{p} - e\boldsymbol{A} + \operatorname{grad} F)^2.$$

To make $\hat{H} = H$ we have to make

$$\operatorname{grad} F = 2e\boldsymbol{A}(\boldsymbol{r}),$$

which for $\boldsymbol{H} \neq 0$ can not be satisfied. The system is therefore irreversible. The same result would of course have been found by considering the Lorentz equations of motion. The discussion of the Hamiltonian is here preferred, because it lends itself more easily to a quantum mechanical generalisation.

1·3. *Transformation of Dynamical Variables.* – Every dynamical variable Q related to a system of particles may be written as a function of the coordinates and the velocities of the particles; in the case of an electromagnetic field forming part of the system, the dynamical variables will be functions of the vector potential (evaluated at the position of the i-th particle) as well. It is easy to see that with the definition of time reversal given in the last section, the velocities $\boldsymbol{r}$ change sign under time reversal. E.g. in the case of the Hamiltonian (3) one has

$$\dot{\boldsymbol{r}} = \boldsymbol{p} + \boldsymbol{r}V(r).$$

Changing $\boldsymbol{p}$ into $-\boldsymbol{p} - 2\boldsymbol{r}V$, $\dot{\boldsymbol{r}}$ changes in $-\dot{\boldsymbol{r}}$. So the kinetic momenta of the particles defined relativistically by

$$\boldsymbol{K} = (1 - \beta^2)^{-\frac{1}{2}} m\dot{\boldsymbol{r}},$$

change sign and the same happens for the angular moments (with respect to a fixed centre) of all the particles.

It follows from these observations that we could have equally well defined reversibility for a system of particles in the following way A system of particles is reversible if there exists an anticanonical transformation which leaves the Hamiltonian invariant and at the same time transforms the kinetic momenta in minus themselves.

2. – Reversibility in Quantum Mechanics.

2·1. *The Matrix K.* – In quantum theory the development of a system is described by the Schrödinger equation

$$i\dot{\psi}(t) = H\psi(t) \,. \tag{9}$$

If the Hamiltonian is time-independent, we can always construct from any given solution $\psi(t)$ of (9) a reversed solution $\hat{\psi}(t)$, which satisfies the Schrödinger equation

$$-i\dot{\hat{\psi}}(t) = H\hat{\psi}(t) \,, \tag{10}$$

which differes from (9) in that the direction of time is reversed.

To show this we observe that H being Hermitian the eigenvalues of H and H^* must be the same (7). In a given representation therefore there must exist a matrix K such that

$$KH^*K^+ = H \,. \tag{11}$$

Putting

$$\hat{\psi}(t) = K\psi^*(t) \,, \tag{12}$$

it is immediately seen that the $\hat{\psi}(t)$ thus defined satisfies equation (10).

If the solutions of the time independent Schrödinger equation

$$H\psi_a = E_a\psi_a \tag{13}$$

are known and if the ψ_a can be expanded in terms of the base φ_α:

$$\psi_a = \sum_\alpha \varphi_\alpha U_{\alpha a} \,,$$

one finds that one possible form for K satisfying equation (11) is

$$K_{\alpha\beta} = \sum_a U_{\alpha a}U_{\beta a} = K_{\beta\alpha} \,. \tag{14}$$

From any given K matrix any other K matrix K' can be obtained by means of

$$K' = KX \,, \tag{15}$$

where X is a unitary operator which commutes with H.

---

(7) The complex conjugate A^* of an operator A is defined by:

$$(\psi_a^* A^* \psi_{a'}) = (\psi_a^* A \psi_{a'})^* .$$

The K's are not operators. If V is the operator which effects a change of base, K transforms according to

$$K' = VKV^{+*}, \tag{16}$$

while the transformation law of an operator Q is $Q' = VQV^{+}$.

We note the antiunitary property of K expressed by (12) or by

$$(\hat{\Psi}^{*}\hat{\Phi}) = (\Phi^{*}\Psi), \tag{17}$$

which holds for two arbitrary wavefunctions.

Since the matrix K always exists for conservative systems its existence cannot of course be made a criterion of reversibility. Such a criterion can only be obtained by subjecting K to further supplementary conditions as will be done in paragraphs 2·5 and in part 3.

2·2. *Reversed Operators and Reversed Equations of Motion.* – In the same way as to every Schrödinger function $\psi(t)$ we can find a reversed function $\hat{\psi}(t)$ which satisfies a Schrödinger equation with the sense of time reversed, it is also always possible to find reversed operators $\hat{Q}$ to every given operator Q which satisfy reversed equations in the Heisenberg representation.

To this end we first observe that the Hamiltonian and any operator representing a dynamical variable of a system can be written as functions of certain basic variables p and q for which the commutation relations are given. In particular H will be of the form

$$H = H(\alpha; p_1 q_1, \ldots) \equiv H(\alpha^{*}; \ldots q_1 p_1). \tag{18}$$

Here α represents a set of complex numerical parameters (as for example the Dirac γ-matrices in the case of the 2nd quantisation of the Dirac equation). The second part of equation (18) is due to the Hermiticity of H. Without loss of generality it can be assumed that the p and q are Hermitian operators, since any complex operator P can be written as $P = p_1 + ip_2$ and the imaginary unit can be included in the set of parameters α.

The reversed operator $\hat{Q}$ of a Hermitian operator Q is defined by

$$\hat{Q} = KQ^{*}K^{+} \tag{19}$$

where, if $Q = Q(\alpha, p_1, q_1, \ldots)$, $Q^{*} = Q(\alpha^{*}, p_1^{*}, q_1^{*}, \ldots)$; K is any particular one of the matrices K defined by equation (11) and discussed in the preceding section.

We shall now consider the equations of motion for Q in the Heisenberg representation.

From

$$\dot{Q} = -i[H, Q] = f(\alpha; p_1 q_1 \ldots) , \tag{20}$$

(the last part of which defines the Hermitian function f) and (19) and (11) we get

$$\hat{\dot{Q}} = i[H, \hat{Q}] . \tag{21}$$

On the other hand, defining as usual $\hat{\dot{Q}}$ by

$$\hat{\dot{Q}} = -i[H, \hat{Q}] \tag{22}$$

we find

$$\hat{\dot{Q}} = -\hat{\dot{Q}} . \tag{23}$$

For the equations of motion we obtain from the 2nd part of (20) and from (23)

$$\hat{\dot{Q}} = -f(\alpha^*; \hat{p}_1 \hat{q}_1 \ldots) = -f(\alpha; \ldots \hat{q}_1 \hat{p}_1). \tag{24}$$

This represents the justification for calling $\acute{Q}$ the reversed operator: it satisfies the same equation of motion as Q, with t replaced by $-t$, all the operators replaced by the reversed operators and their order inverted. Instead of inverting the order of the operators we can also retain the order and replace all the numerical coefficients in the equation of motion by their complex conjugates.

For the matrix elements of the reversed operators defined by (19) one has identically:

$$(\Psi^* Q \Phi) = (\hat{\Phi}^* \hat{\acute{Q}} \hat{\Psi}) \tag{25}$$

provided that Q is Hermitian.

Before continuing with the general line of argument we should like to comment on the extension of equations (19), (23) and (25) to the case of non Hermitian operators:

$$X = Q_1 + iQ_2 \qquad\qquad Q_i = \text{Hermitian} . \tag{26}$$

If we want to retain equation (19) also for non Hermitian operators we have

$$\hat{X} = KX^*K^+ = \hat{Q}_1 - i\hat{\acute{Q}}_2 \tag{27}$$

and instead of (25) we obtain

$$(\Psi^* X \Phi) = (\hat{\Phi}^* \hat{X}^+ \hat{\Psi}) . \tag{28}$$

On the other hand we can also define the reversed of a complex operator in such a way that equation (25) remains valid. In this case we have

$$\hat{X} = \hat{Q}_1 + i\hat{Q}_2 \tag{29}$$

and instead of (19)

$$\hat{X}^+ = KX^*K^+ . \tag{30}$$

The equivalence of the last two members of equation (24) does no longer hold since f is no longer a Hermitian operator. In the case of definition (27) we have for the reversed of the equation of motion

$$\dot{X} = f(\alpha; X_1 X_2 \ldots)$$

the equation

$$\hat{\dot{X}} = -f(\alpha^*; \hat{X}_1 \hat{X}_2 \ldots) . \tag{31}$$

For definition (29) we have on the other hand

$$\hat{\dot{X}} = -f(\alpha; \ldots \hat{X}_2 \hat{X}_1) . \tag{32}$$

The difference between the definitions (27) and (29) is purely formal: all the physically observable operators are Hermitian and for Hermitian operators the two definitions coincide. In the detailed discussion of particular systems we shall adopt definition (29).

2·3. *Transition Amplitudes.* – The formalism developed so far and characterised by the absence of a closer specification for the matrix K, already allows one to treat the reciprocity theorem, mentioned in the introduction.

For this we consider the transition operator $S(t - t_0)$ defined by

$$S(t - t_0) = I - i \int_{t_0}^{t} HS(t' - t_0)\, \mathrm{d}t'$$

for the solution of which we may write symbolically

$$S(t - t_0) = \exp[-iH(t - t_0)]. \tag{33}$$

This operator serves to determine the transition amplitude corresponding to a transition from a state Ψ at time t_0 to a state Φ at time t:

$$(\Phi^* S(t - t_0)\Psi) .$$

With either definition (27) or (29) given in the previous section one imme-

diately obtains

$$(\Phi^* S(t-t_0)\Psi) = (\hat{\Psi}^* S(t-t_0)\hat{\Phi}) \,. \tag{34}$$

This equation means that the transition amplitude for a transition from a state Ψ to a state Φ in the time-interval $t - t_0$ is equal to the transition amplitude for a transition from a state $\hat{\Phi}$ to a state $\hat{\Psi}$ in the same time interval. If K is known this relation can be used to reduce the number of independent elements of the transition operator. In general, however, the determination of K may be tantamount to solving the Schrödinger equation of the problem, so that the usefulness of the reciprocity theorem is in effect based on the possibility of finding a K without actually solving the Schrödinger equation. This is the case in all time reversible systems as we shall see. But in principle there could be irreversible systems for which a K is known (which satisfies (11) but not the supplementary conditions which we shall introduce later) and the theorem would still hold in these cases.

2·4. *An Example.* – To illustrate the formalism developed so far let us consider the case of a particle moving in an external magnetic field with axial symmetry. This system is irreversible in classical theory, if the electromagnetic field is not reversed together with t. Yet a matrix K can be defined in this case.

The Hamiltonian of the system is

$$H = \frac{1}{2mr^2}(p_\varphi - erA_\varphi) + \frac{p_r^2}{2m} + V(r). \tag{35}$$

Here cylindrical coordinates have been used and the motion is studied only in the plane $z = 0$. By inspection of the Schrödinger equation associated with (35) one finds that a complete set of solutions can be given in the form

$$\psi_{n,m}(r;\varphi) = R_{n,m}(r) \exp[im\varphi] \,. \tag{36}$$

Since the radial part of the Schrödinger equation does not contain the imaginary unit, $R_{n,m}$ can be chosen to be real. Using equation (14) we find that in the r-φ-representation K has the form

$$K_{r\varphi,r'\varphi'} = \delta(r-r')\delta(\varphi+\varphi') \,, \tag{37}$$

which together with equation (19) gives the following relations between the reversed and not reversed variables:

$$\begin{cases} \hat{r} = r & \hat{p}_r = -p_r \,, \\ \hat{\varphi} = -\varphi & \hat{p}_\varphi = p_\varphi \,. \end{cases} \tag{38}$$

It is therefore seen that $\hat{\boldsymbol{r}} \neq \boldsymbol{r}$ (8).

2·5. *The Reversibility Condition for Quantised Systems.* – The symmetry property discussed in the previous sections is common to all conservative systems and although it may in certain cases be useful for the limitation of the transition matrix it can certainly not be used to exclude certain forms of conservative Hamiltonians or interactions.

To define time reversibility in quantum mechanics we shall proceed in analogy with the classical case. We first observe that the transformation (19) leads from the commutation relations

$$[p_i, q_k] = -i\delta_{ik} \qquad [q_i, q_k] = 0 \qquad [p_i, p_k] = 0\,, \tag{39}$$

to the commutation relations for the reversed operators

$$[\hat{p}_i, \hat{q}_k] = +i\delta_{ik} \qquad [\hat{q}_i, \hat{q}_k] = 0 \qquad [\hat{p}_i, \hat{p}_k] = 0\,. \tag{40}$$

The change in the sign of i corresponds to the change of sign in the Poissson brackets under time reversal in classical theories and represents the analogue of the anticanonical transformations discussed in section 1·2. In the classical case we have defined time reversibility by considering those anticanonical transformations which leave the position variables unaltered or change the sign of the kinetic momenta. We shall in this section do the same for a system of quantised point-particles; in the next section we shall consider the case of field theory. A system of spinless particles we shall call reversible if there exists a matrix K satisfying equation (11) and

$$K\boldsymbol{r}_i^* K^+ = \boldsymbol{r}_i\,. \tag{41}$$

It has already been remarked (8) in section 4 that such a definition labels the system treated in the previous section as irreversible. In the case of spinning particles we need an additional requirement regulating the transformation of the spin operators; the correspondence principle and the fact that the spin operator $\boldsymbol{\sigma}$ has the character of an angular momentum, leads us to require that for a system to be reversible K in addition to (11) and (41) must also satisfy the condition

$$K\boldsymbol{\sigma}^{(i)*} K^+ = -\boldsymbol{\sigma}^{(i)}. \tag{42}$$

(8) If $\hat{\boldsymbol{r}} = \boldsymbol{r}$ were imposed as a supplementary condition on K the system described by the Hamiltonian (35) would not be reversible just as in the classical case. It is indeed a condition of this type which we shall impose in the next section in order to distinguish between reversible and irreversible systems.

This exhausts the case of non relativistic quantised particle mechanics, apart from the isotopic spin formalism which does not require separate treatment.

3. – Quantised Fields.

3·1. *Spin* 0 *Particles.* – In this chapter we want to discuss the restrictions which the correspondence principle applied to the theory of fields imposes on the matrix K.

In addition to the Hamiltonian of a field we also have to know the physical significance of the variables from which the Hamiltonian is formed; this means that a prescription has to be given for the physical interpretation of the quantities describing the system. Usually such physically significant quantities, which have to be defined in all field theories, are the operators of the occupation numbers, which determine the number of particles in a given state.

In the case of spinless neutral Bosons the state of a particle is completely defined by giving its kinematic momentum $\boldsymbol{k}$. If $N_{\boldsymbol{k}}$ is the operator of the occupation number of a state with momentum $\boldsymbol{k}$, the obvious condition suggested by the correspondence principle is

$$KN_{\boldsymbol{k}}^{*}K^{+} = N_{-\boldsymbol{k}} = \hat{N}_{\boldsymbol{k}} . \tag{43}$$

In fact there will always exist a set of states Ψ in which the behaviour of the system is classical and for these states we must have

$$(\Psi^{*}\hat{N}_{\boldsymbol{k}}\Psi) = (\Psi^{*}N_{-\boldsymbol{k}}\Psi) . \tag{44}$$

Equation (43) is only an extrapolation of (44) in so far as one has to assume that (44) is valid not only for the « quasiclassical states » Ψ but for every state of the system.

We now have to examine to what extent the K-matrix is determined by condition (43). With creation and annihilation operators defined as in WENTZEL [9] we have

$$\Phi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2\omega_{\boldsymbol{k}}}} (a_{\boldsymbol{k}} + a_{-\boldsymbol{k}}^{+}) \exp [i\boldsymbol{k}\cdot\boldsymbol{r}) , \tag{45}$$

where $\Phi(\boldsymbol{r})$ is the operator of the meson field, $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^{+}$ have to satisfy the commutation relations

$$[a_{\boldsymbol{k}}^{+}, a_{\boldsymbol{k}'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'} \qquad [a_{\boldsymbol{k}}^{+}, a_{\boldsymbol{k}'}^{+}] = 0 . \tag{46}$$

[9] G. WENTZEL: *Quantum Theory of Fields* (New York, 1949).

Using equation (30) one gets for the reversed commutation relations

$$[\hat{a}_{\boldsymbol{k}}^{+}, \hat{a}_{\boldsymbol{k}'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'} \qquad [\hat{a}_{\boldsymbol{k}}^{+}, \hat{a}_{\boldsymbol{k}'}^{+}] = 0\,, \tag{47}$$

of which a particular solution is obviously

$$\hat{a}_{\boldsymbol{k}}^{+} = a_{-\boldsymbol{k}}\,. \tag{48}$$

This solution also satisfies the condition (43), if we remember that

$$N_{\boldsymbol{k}} = a_{\boldsymbol{k}}^{+} a_{\boldsymbol{k}}\,.$$

We have to find the most general solution of (47) and (43). From any particular set of variables satisfying a given set of commutation relations, the most general set of variables can be obtained by performing a unitary transformation. In order that (43) be satisfied, the operator of the unitary transformation can only depend on all the $N_{\boldsymbol{k}}$, (i.e. must be of the form

$$\exp iF(N_{\boldsymbol{k}_1}, N_{\boldsymbol{k}_2}, \ldots) \equiv \exp iF[N]\,,$$

so that the most general solution has the form

$$\hat{a}_{\boldsymbol{k}} = \exp iF[N] a_{-\boldsymbol{k}}^{+} \exp -iF[N]\,. \tag{49}$$

The transformation law for the function Φ becomes

$$\hat{\Phi}(\boldsymbol{r}) = \exp iF[N]\Phi(\boldsymbol{r}) \exp -iF[N]\,, \tag{50}$$

and it is seen that this in general will not be a local transformation. We note for the following two very particular cases of the function F:

1) the trivial case $F = 0$ gives

$$\hat{\Phi}(\boldsymbol{r}) = \Phi(\boldsymbol{r})\,; \tag{51}$$

2) for $F = \pi \sum_{\boldsymbol{k}} N_{\boldsymbol{k}}$ one obtains

$$\hat{\Phi}(\boldsymbol{r}) = -\Phi(\boldsymbol{r})\,. \tag{52}$$

In the case of charged spin 0 Bose fields the occupation numbers are of the form $N_{\boldsymbol{k},\varepsilon}$ ($\varepsilon = \pm 1$) and the supplementary condition on K suggested by the correspondence principle is

$$\hat{N}_{\boldsymbol{k},\varepsilon} = N_{-\boldsymbol{k},\varepsilon}\,. \tag{53}$$

The most general transformation of the field operator Φ is

$$\hat{\Phi}^{\vdash}(\boldsymbol{r}) = \exp iF[N_+, N_-]\Phi(\boldsymbol{r}) \exp -iF[N_+, N_-] . \tag{54}$$

Generally for any combination of spinless Boson fields, also describing Bosons with different masses, the obvious generalisation of equation (43) is

$$\hat{N}^{(i)}_{\boldsymbol{k}} = N^{(i)}_{-\boldsymbol{k}} , \tag{55}$$

where the index i labels the charge and the mass of the particles.

3·2. *The Maxwell Field.* – The state of a photon is specified not only by its kinetic momentum but also by its state of polarization. The latter will be described by a variable s which can assume the values ± 1 corresponding respectively to right and left polarized photons. Following WENTZEL we may introduce creation and annihilation operators $a^{\vdash}_{\boldsymbol{k}+}$, $a^{+}_{\boldsymbol{k}-}$, $a_{\boldsymbol{k}+}$, $a_{\boldsymbol{k}-}$ for the two types of photon. The occupation numbers are then defined by

$$N_{\boldsymbol{k},s} = a^{+}_{\boldsymbol{k},s} a_{\boldsymbol{k},s} , \tag{56}$$

and represent respectively the numbers of right ($s = +1$) and left ($s = -1$) polarised photons with momentum $\boldsymbol{k}$. The operator of the transverse part of the vector potential then takes the form

$$\boldsymbol{A}(\boldsymbol{r}) = \sum_{\boldsymbol{k},s} \sqrt{\frac{1}{2|\boldsymbol{k}|}} \, \boldsymbol{c}_{\boldsymbol{k},s} \, (a_{\boldsymbol{k},s} + a^{+}_{-\boldsymbol{k},s}) \exp [i\boldsymbol{k}\cdot\boldsymbol{r}] , \tag{57}$$

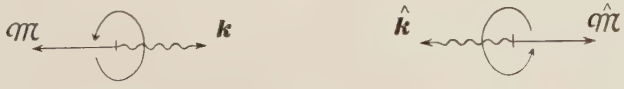
where the $\boldsymbol{c}_{\boldsymbol{k}\pm}$ are unitary null vectors defined by

$$\boldsymbol{c}_{\boldsymbol{k}\pm} = \frac{1}{\sqrt{2}} (\boldsymbol{e}^{(2)}_{\boldsymbol{k}} \pm i\boldsymbol{e}^{(3)}_{\boldsymbol{k}}) , \tag{58}$$

$\boldsymbol{e}^{(1)}_{\boldsymbol{k}}$, $\boldsymbol{e}^{(2)}_{\boldsymbol{k}}$, $\boldsymbol{e}^{(3)}_{\boldsymbol{k}}$ form a right hand system with $\boldsymbol{e}^{(1)}_{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$. From the classical analogy it follows that the intrinsic angular moment of the photon should be reversed under time reversal. Since the $\boldsymbol{k}$-component of the angular momentum is $+\hbar$ for a right polarised photon and $-\hbar$ for a left polarised photon, we shall postulate in generalisation of (43)

$$\hat{N}_{\boldsymbol{k},s} = N_{-\boldsymbol{k},s} . \tag{59}$$

That this really reverses the angular momentum $\mathcal{M}$ can be seen from the following diagrams



A possible solution of this equation is

$$\hat{a}^{+}_{\boldsymbol{k},s} = a_{-\boldsymbol{k},s}\,, \tag{60}$$

and the most general solution

$$\hat{a}^{+}_{\boldsymbol{k},s} = \exp iF[N]a_{-\boldsymbol{k},s}\exp -iF[N]\,, \tag{61}$$

where $F[N] = F(N_{\boldsymbol{k}_1+}, N_{\boldsymbol{k}_2+}, \ldots; N_{\boldsymbol{k}_1-}, \ldots)$. The most general solution for A is

$$\hat{\boldsymbol{A}}(\boldsymbol{r}) = \exp iF[N]\boldsymbol{A}(\boldsymbol{r})\exp -iF[N]\,, \tag{62}$$

where again we may have as special cases

$$\hat{\boldsymbol{A}}(\boldsymbol{r}) = \pm\,\boldsymbol{A}(\boldsymbol{r})\,, \tag{63}$$

according as $F = 0$ or $F = \pi\sum_{\boldsymbol{k},s} N_{\boldsymbol{k},s}$.

3·3. *The Dirac Field.* – In this section we consider Dirac particles with non vanishing restmass. They are described by a field operator $\psi(\boldsymbol{r})$, which can be expanded in the form

$$\psi(\boldsymbol{r}) = \sum_{\boldsymbol{k},s,\varepsilon} u(\boldsymbol{k}, s, \varepsilon)a(\boldsymbol{k}, s, \varepsilon)\exp i\boldsymbol{k}\cdot\boldsymbol{r}\,. \tag{64}$$

The a are creation operators (for $\varepsilon = -1$) and annihilation operators (for $\varepsilon = +1$) and the u's are the simultaneous solutions of the equations

$$\begin{cases} [(\boldsymbol{\alpha}\cdot\boldsymbol{k}) + \beta M - \varepsilon E]u(\boldsymbol{k}, s, \varepsilon) = 0 \\ (\boldsymbol{\sigma}\cdot\boldsymbol{k})u(\boldsymbol{k}, s, \varepsilon) = |\boldsymbol{k}|su(\boldsymbol{k}, s, \varepsilon); \end{cases} \tag{65}$$

their phases are fixed by the condition [10]

$$u^*(\boldsymbol{k}, s, \varepsilon) = u(-\boldsymbol{k}, s, -\varepsilon)\,, \tag{66}$$

$E = +\sqrt{M^2 + k^2}$ is the energy and $\sigma_1 = i\alpha_2\alpha_3$, ... is the spin operator. $\varepsilon = \pm 1$ refers to positive and negative energy states respectively and $s = \pm 1$ (as in the case of photons) corresponds to an orientation of the spin parallel or antiparallel to the momentum.

The operators $a(\boldsymbol{k}, s, \varepsilon)$ satisfy the anticommutation relations

$$\left\{a^{+}(\boldsymbol{k}, s, \varepsilon), a(\boldsymbol{k}', s', \varepsilon')\right\} = \delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{ss'}\delta_{\varepsilon\varepsilon'}; \qquad \left\{a(\boldsymbol{k}s\varepsilon), a(\boldsymbol{k}'s'\varepsilon')\right\} = 0\,. \tag{67}$$

[10] We use the Majorana Gauge: $\beta^* = -\beta$, $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}$.

The number of Dirac particles with momentum $\boldsymbol{k}$ and spin s is given by

$$N_{\boldsymbol{k},s,+} = a^+(\boldsymbol{k}, s, +)a(\boldsymbol{k}, s, +)\,, \tag{68}$$

and the number of antiparticles by

$$N_{\boldsymbol{k},s,-} = a(-\boldsymbol{k}, s, -)a^+(-\boldsymbol{k}, s, -)\,. \tag{69}$$

In complete analogy with the photon case we conclude from the correspondence principle

$$\hat{N}_{\boldsymbol{k},s,\varepsilon} = N_{-\boldsymbol{k},s,\varepsilon}\,. \tag{70}$$

This can in particular be satisfied by putting

$$\hat{a}^+(\boldsymbol{k}, s, \varepsilon) = a(-\boldsymbol{k}, s, \varepsilon)\,. \tag{71}$$

Before considering the most general form of the transformation law between the $a(\boldsymbol{k}, s, \varepsilon)$ we want to see how $\psi(\boldsymbol{r})$ transforms under the substitution (71). Using equations (30), (71) and (66):

$$\begin{aligned}\hat{\psi}^+(\boldsymbol{r}) &= \sum_{\boldsymbol{k},s,\varepsilon} \hat{a}^+(\boldsymbol{k}, s, \varepsilon)u^*(\boldsymbol{k}, s, \varepsilon) \exp -i\boldsymbol{k}\cdot\boldsymbol{r} \\ &= \sum_{\boldsymbol{k},s,\varepsilon} a\ (\boldsymbol{k}, s, \varepsilon)\ u(\boldsymbol{k}, s, -\varepsilon) \exp i\boldsymbol{k}\cdot\boldsymbol{r}\,.\end{aligned} \tag{72}$$

We observe further that with

$$U = i\beta\alpha_1\alpha_2\alpha_3\,, \tag{73}$$

one has

$$Uu(\boldsymbol{k}, s, \varepsilon) = u(\boldsymbol{k}, s, -\varepsilon)\,. \tag{74}$$

U is one particular form of the operator which is commonly used to represent time reflection. Inserting into (72) we then obtain

$$\hat{\psi}^+(\boldsymbol{r}) = U\psi(\boldsymbol{r})\,. \tag{75}$$

This form of the transformation corresponds to the special choice (71). The most general solution of (70) is of the form

$$\hat{a}^+(\boldsymbol{k}, s, \varepsilon) = \exp iF[N]a(-\boldsymbol{k}, s, \varepsilon) \exp -iF[N]\,. \tag{76}$$

The four « types » of Fermions represent special cases. According as $F(N) = 0$,

or $= (\pi/2,\ \pi,\ 3\pi/2) \sum_{\boldsymbol{k}\, s,\varepsilon} N_{\boldsymbol{k},s,\varepsilon}$ one has the transformation law

$$\hat{\psi}^{+}(\boldsymbol{r}) = \alpha U \psi(\boldsymbol{r}) \qquad \text{with } \alpha = 1, i, -1, -i\,. \tag{77}$$

It must be emphasized that these types can only become meaningful, if interactions are drawn into consideration.

3·4. *General Definition of Reversibility in Field Theory.* – We are now in a position to give a general definition of reversibility in field theory. For this we assume that the state of the field (or of an assembly of interacting fields) may be completely specified by a set of occupation numbers $\mathcal{N}_Q$ [11]. The indices Q which define the one particle states of the system — as the momentum $\boldsymbol{k}$, the spin, the charge, etc. of a particle — have a classical analogue which under time reversal transforms in a definite way: $Q \to \tilde{Q}$, say. The system will now be called reversible if there exists an antiunitary transformation, which leaves the Hamiltonian invariant and transforms $\mathcal{N}_Q$ in $\mathcal{N}_{\tilde{Q}}$; in other words if there exists a matrix K such that

$$\begin{cases} KH^{*}K^{+} = H \\ K\mathcal{N}_Q^{*}K^{+} = \mathcal{N}_{\tilde{Q}}\,, \end{cases} \tag{78}$$

holds. The second condition (78) is — as in the case of the examples discussed in the preceding sections — based in the correspondence principle.

3·5. *Criterion for Non Reversibility.* – To decide whether a system is reversible or not one has to explore the whole class of transformations satisfying equations (78). As we have already seen, this class is very much wider than the one usually considered and it will therefore be in general be quite difficult to exclude a system on the grounds of irreversibility.

In this section therefore, we want to discuss the condition for irreversibility. To this end we shall consider a Hamiltonian of the form

$$H = H_0 + H_1\,, \tag{79}$$

in which H_0 represents the Hamiltonian of the free fields (and is therefore a function of the $\mathcal{N}_Q$ alone) and H_1 contains all the interactions between the fields. On account of the 2nd equation (79) H_0 is transformed into itself. Let us now consider one particular transformation K_0 which transforms H into

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[11] A word is necessary to clarify the meaning of the various N and $\mathcal{N}$ introduced; by N (with some indices and suffixes) we mean the occupation numbers of one field; by $\mathcal{N}$ we mean the occupation numbers describing the state of several fields; e.g. in the case of two fields to give the $\mathcal{N}$'s means to give two sets of N's.

itself. If $K_0 H_1^* K_0^+ = H_1$ then the system is certainly reversible. Let us therefore assume that

$$K_0 H_1^* K^+ = \tilde{H}_1 \neq H_1 \,. \tag{80}$$

In order that the system be reversible there must therefore exist a function $F[\mathcal{N}]$ of all the occupation numbers such that

$$\exp iF[\mathcal{N}]\tilde{H}_1 \exp - iF[\mathcal{N}] = H_1 \,. \tag{81}$$

In the representation in which the occupation numbers are diagonal this equation reads

$$\langle \mathcal{N}'' | \tilde{H}_1 | \mathcal{N}' \rangle \exp i(F[\mathcal{N}''] - F[\mathcal{N}']) = \langle \mathcal{N}'' | H_1 | \mathcal{N}' \rangle \,, \tag{82}$$

where $\langle \mathcal{N}''|$ and $|\mathcal{N}'\rangle$ denote any two arbitrary states characterised by the sets of occupation numbers $\mathcal{N}''$ and $\mathcal{N}$ respectively. If one can show that for no choice of the function F eq. (82) can be satisfied for all the pairs $\mathcal{N}''$, $\mathcal{N}'$ one has shown that the system is irreversible. This is our criterion for irreversibility. Note that if the modulus of the ratio

$$_{\mathcal{N}''}R_{\mathcal{N}'} = \frac{\langle \mathcal{N}'' | \tilde{H}_1 | \mathcal{N}' \rangle}{\langle \mathcal{N}'' | H_1 | \mathcal{N}' \rangle} \tag{83}$$

is different from one for at least a pair of states $\langle \mathcal{N}''|$, $|\mathcal{N}'\rangle$ equation (83) can certainly not be satisfied for any choice of F and the system is surely irreversible. However this is just a sufficient condition for time irreversibility and hence also if the above ratio is one the system may not be reversible as will be the case in an example considered below ([12]). Examples of time irreversible systems in the theory of fields may be given, although, as it is apparent from what has been said, the assertion that a given system is time irreversible implies a much longer analysis than usually given.

We shall discuss in detail just one of them and show in the next section that a neutral boson field scalarly and vectorially coupled to a Dirac field is time irreversible.

3·6. *An example of an irreversible system.* – We consider the interaction energy:

$$H_1 = \left[\int g\psi^+\beta\psi\Phi \, \mathrm{d}\tau + f\int\!\!\int \{ \psi^+\psi\pi + (\psi^+\boldsymbol{\alpha}\psi)\cdot \mathrm{grad}\, \Phi \, \mathrm{d}\tau \} \right] = \mathcal{H}_1 + \mathcal{H}_2 \,. \tag{84}$$

([12]) A case in which the condition $|_{\mathcal{N}''}R_{\mathcal{N}'}| = 1$ may be useful in deriving limitations on the interaction is the one considered by H. A. Tolhoek and S. R. De Groot: (*Phys. Rev.*, **84**, 151 (1951)) and by L. C. Biedenharm and M. R. Rose (*Phys. Rev.*, **83**, 459 (1951)): a general combination of the several β interactions with complex coefficients.

In the $\mathcal{T}$ representation both $\mathcal{H}_1$ and $\mathcal{H}_2$ have non vanishing matrix elements between the same states. Under the transformation

$$\hat{\Phi}(\boldsymbol{r}) = \Phi(\boldsymbol{r}) \qquad \hat{\Psi}^+(\boldsymbol{r}) = U\Psi(\boldsymbol{r}) .$$

$\mathcal{H}_1$ changes into itself and $\mathcal{H}_2$ into minus itself. Therefore

$$(85) \qquad {}_{\mathcal{T}''}R_{\mathcal{T}'} = \frac{\langle \mathcal{T}'' | \mathcal{H}_1 | \mathcal{T}' \rangle + \langle \mathcal{T}'' | \mathcal{H}_2 | \mathcal{T}' \rangle}{\langle \mathcal{T}'' | \mathcal{H}_1 | \mathcal{T}' \rangle - \langle \mathcal{T}'' | \mathcal{H}_2 | \mathcal{T}' \rangle}$$

Although ${}_{\mathcal{T}''}R_{\mathcal{T}'}$ is in general different from one, its modulus is always one for any $\langle \mathcal{T}''|$ and $|\mathcal{T}'\rangle$ as we shall show in a moment. So we cannot conclude from the consideration of the modulus of ${}_{\mathcal{T}''}R_{\mathcal{T}'}$ that the system is irreversible; for deriving this conclusions we must show, following (82) that for no choice of $F[\mathcal{T}]$ it is possible to have

$$(86) \qquad {}_{\mathcal{T}''}R_{\mathcal{T}'} = \exp i(F[\mathcal{T}''] - F[\mathcal{T}']) .$$

Let us begin by calculating ${}_{\mathcal{T}''}R_{\mathcal{T}'}$.

The states $\langle \mathcal{T}''|$ and $|\mathcal{T}'\rangle$ between which $\mathcal{H}_1$ and $\mathcal{H}_2$ have non vanishing matrix elements may be more explicitely written:

$$\langle \boldsymbol{p}''s''\varepsilon''n''_{\boldsymbol{k}}\nu''| \qquad \text{and} \qquad |\nu' n'_{\boldsymbol{k}}\boldsymbol{p}'s'\varepsilon'\rangle .$$

In this notation $\boldsymbol{p}'s'\varepsilon'$ and $\boldsymbol{p}''s''\varepsilon''$ denote the initial and final momentum, spin and energy of the fermion which changes its state, ν' and ν'' the numbers of all the other fermions, $n'_{\boldsymbol{k}}$ and $n''_{\boldsymbol{k}}$ the numbers of mesons.

One has:

$$n''_{\boldsymbol{k}} = n'_{\boldsymbol{k}} \qquad (\boldsymbol{k} \neq \boldsymbol{p}'' - \boldsymbol{p}')$$

$$n''_{\boldsymbol{p}''-\boldsymbol{p}'} = n''_{\boldsymbol{p}''-\boldsymbol{p}'} + 1$$

and:

$$\nu'' = \nu' .$$

It is easy to check that:

$$(87) \qquad R_{\boldsymbol{p}''s''\varepsilon''n''_{\boldsymbol{k}}\nu'' \leftarrow \boldsymbol{p}'s'\varepsilon'n'_{\boldsymbol{k}}\nu'} = \frac{g(u^*(\boldsymbol{p}''s''\varepsilon'')\beta u(\boldsymbol{p}'s'\varepsilon')) + if(u^*(\boldsymbol{p}''s''\varepsilon'')u(\boldsymbol{p}'s'\varepsilon'))\Delta(\boldsymbol{p}'\boldsymbol{p}'')}{g(u^*(\boldsymbol{p}''s''\varepsilon'')\beta u(\boldsymbol{p}'s'\varepsilon')) - if(u^*(\boldsymbol{p}''s''\varepsilon'')u(\boldsymbol{p}'s'\varepsilon'))\Delta(\boldsymbol{p}'\boldsymbol{p}'')} ,$$

where:

$$\Delta(\boldsymbol{p}'\boldsymbol{p}'') = \omega_{\boldsymbol{p}''-\boldsymbol{p}'} + E_{\boldsymbol{p}'} - E_{\boldsymbol{p}''} \equiv \Delta$$

and:

$$\omega_k = \sqrt{k^2 + \mu^2} \qquad E_p = \sqrt{p^2 + M^2} \qquad \mu = \text{meson mass}$$

$$M = \text{nucleon mass}$$

It is sufficient for the present purpose to calculate R between states with the same values of s and ε.

$$s'' = s' - s\,, \qquad \varepsilon'' = \varepsilon' - \varepsilon\,.$$

In this case one has, whatever the values of s and ε:

$$\underset{\boldsymbol{p}''s''\varepsilon''n''_{\boldsymbol{k}}\nu'' \leftarrow \boldsymbol{p}'s'\varepsilon'n'_{\boldsymbol{k}}\nu'}{R} = \frac{g\Lambda_- + if\Lambda_+\Delta}{g\Lambda_- - if\Lambda_+\Delta}\,, \tag{88}$$

where:

$$\Lambda_\pm = 1 \pm \frac{p'\,p''}{(E_{p'} + M)(E_{p''} + M)}\,,$$

The modulus of expression (88) is one, so that, as we have already seen, we cannot conclude that the system is time irreversible by the argument of the modulus. We must consider equation (86) and show that it cannot be satisfied for any choice of $F[\mathcal{T}]$. The important point in this respect is that expression (88) for R does not depend upon the numbers $n_{\boldsymbol{k}}$ and ν; we shall now show that, as a consequence of this fact, no function $F[\mathcal{T}]$ satisfying (86) can be found.

Let us write $F[\mathcal{T}]$ as:

$$F[\mathcal{T}] = \sum_{\boldsymbol{p}s\varepsilon} a(\boldsymbol{p}s\varepsilon) n_{\boldsymbol{p}s\varepsilon} + \sum_{\boldsymbol{k}} b(\boldsymbol{k}) n_{\boldsymbol{k}} + \sum_{\boldsymbol{p}s\varepsilon} c(\boldsymbol{p}s\varepsilon, \boldsymbol{k}) n_{\boldsymbol{p}s\varepsilon} n_{\boldsymbol{k}} +$$
$$+ \sum_{\boldsymbol{p}s\varepsilon, \bar{\boldsymbol{p}}\bar{s}\bar{\varepsilon}} d(\boldsymbol{p}s\varepsilon, \bar{\boldsymbol{p}}\bar{s}\bar{\varepsilon}) n_{\boldsymbol{p}s\varepsilon} n_{\bar{p}\bar{s}\bar{\varepsilon}} + \sum_{\boldsymbol{k},\boldsymbol{k}} f(\boldsymbol{k}, \boldsymbol{k}) n_{\boldsymbol{k}} n_{\boldsymbol{k}} + \text{higher order terms},$$

where $n_{\boldsymbol{p}s\varepsilon}$ denote the number of fermions and $n_{\boldsymbol{k}}$ the number of bosons. The difference $F[\mathcal{T}''] - F[\mathcal{T}']$ must not depend on the $n_{\boldsymbol{k}}$ and ν whatever the values of $\boldsymbol{p}''$ and $\boldsymbol{p}'$ are. This implies that in the above difference the coefficients multiplying all the terms of order higher than one in the $n_{\boldsymbol{p}s\varepsilon}$ and n_k must vanish. So in order that equation (86) be satisfied we must have:

$$a(\boldsymbol{p}''s\varepsilon) - a(\boldsymbol{p}'s\varepsilon) - b(\boldsymbol{p}' - \boldsymbol{p}'') = \Gamma(\boldsymbol{p}', \boldsymbol{p}'')\,, \tag{89}$$

where $\Gamma(\boldsymbol{p}', \boldsymbol{p}'')$ is defined by:

$$R = \exp i\Gamma\,.$$

One may immediately realize that in the limit: $\boldsymbol{p}'' \to \boldsymbol{p}'$ the right side of equation (89) still depends on $\boldsymbol{p}'$ while the left side is independent from $\boldsymbol{p}'$. Hence equation (89) cannot be satisfied and the system considered is time irreversible.

3·7. *Some remarks.* – We want to make some remarks with respect to the questions which have been discussed and to some which still remain to be examined.

Our main line of argument has been to introduce time reversibility into q.m. by trying to preserve as many as possible of the symmetry properties of classical systems with which we are familiar. It is by all means possible that by our definition of time reversal we have not exhausted all the information which could be drawn from the correspondence principle. Our point is however that although more restrictive definitions of time reversal might perhaps be given, the definition here proposed is already sufficient to classify some systems as time irreversible in an unambiguous way.

An examination of other systems will be the object for future work.

Another point which still needs discussion is to see to what physical inconsistences, if any, would lead a system classified as time irreversible; surely if we assume that a time reversible system may be a physically significant one the correspondence principle should be violated; but would this give rise to contradiction with macroscopic esperience?

Finally a word may be said concerning our extensive use of the occupation number operators. Our only point in this respect is that, although these operators are the numbers of the bare particles, (as opposite to physical particles), these quantities in the present stage of quantum field theory play a dominant role in the physical interpretation of the theory.

RIASSUNTO

Si rileva che una definizione soddisfacente della operazione di inversione del tempo in meccanica quantistica non è ancora stata data, così che, per esempio, le ragioni per escludere un sistema in quanto irreversibile sono alquanto oscure. In questo lavoro vien data una tale definizione, tenendo anche conto del fatto che uno stesso sistema può essere descritto da tutta una classe di Hamiltoniane equivalenti. Una tale definizione conseguentemente esclude sistemi irreversibili, mentre le vecchie prescrizioni escludevano una particolare Hamiltoniana, della classe di quelle descriventi uno stesso sistema. La definizione qui proposta è suggerita dal principio di corrispondenza ed è basata completamente sulle proprietà di trasformazione di osservabili. La nuova definizione è sufficiente ad escludere, come viene mostrato, alcuni sistemi le cui Hamiltoniane erano già state classificate irreversibili, sebbene il metodo di dimostrazione, a causa della maggior generalità, sia alquanto più complicato dell'usuale.

Introduzione di una lunghezza fondamentale nella teoria classica dell'elettrone.

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Riassunto. — Dopo un esame delle difficoltà inerenti alle teorie classiche dell'elettrone in interazione col campo elettromagnetico, con particolare riguardo alle teorie di Abraham-Lorentz e di Dirac-Eliezer, viene proposta attraverso l'introduzione di una lunghezza fondamentale avente il carattere di costante universale, una equazione alle differenze finite, relativisticamente invariante, che sembra in grado di superare le principali delle suddette difficoltà. Si mostrano alcune sue applicazioni a problemi semplici e si imposta una formulazione generale lagrangiana. Vengono poi stabilite le connessioni di tale teoria con altre (particelle polo-dipolari, ecc., campi a interazione non locale) e se ne studia l'approssimazione non relativistica che viene a coincidere con modelli più volte proposti di elettrone esteso. Infine si mostra come tale teoria possa essere inquadrata in una visione generale del campo elettromagnetico e dell'interazione di esso con le proprie sorgenti. La massa dell'elettrone è qui considerata di natura elettromagnetica. Come ulteriore interessante conseguenza viene esaminata la possibilità di una interpretazione classica del momento magnetico anomalo dell'elettrone.

1. – Generalità.

Uno dei problemi fondamentali che si presentano nella odierna teoria dei campi, considerata da un punto di vista classico, è quello di dedurre le equazioni del moto delle particelle, che costituiscono le sorgenti del campo, dalle equazioni stesse del campo.

Nel caso dell'elettrodinamica tale problema è stato affrontato fin da mezzo secolo fa. Abraham fu il primo ad avanzare l'idea che la massa dell'elettrone fosse interamente di natura elettromagnetica nel senso che, conformemente

alla teoria di Maxwell, un elettrone in moto produce un campo elettromagnetico a cui è associata una certa energia (e una certa quantità di moto) che si manifesta sotto forma di inerzia dell'elettrone, cosicchè ad ogni variazione della velocità di questo corrisponde una variazione del campo associato. La teoria di Abraham, che assumeva un modello di elettrone rigido di forma sferica, ha dovuto però essere abbandonata per le gravi difficoltà cui andava incontro.

Maggior successo ebbe invece la teoria di Lorentz, che introduceva l'ipotesi di un elettrone sferico a riposo ma contratto in moto, lungo la direzione del moto stesso, nel rapporto $1/\sqrt{1-\beta^2}$, essendo $\beta = v/c$, con v velocità dell'elettrone. Però anche la teoria di Lorentz è tutt'altro che scevra di difficoltà, soprattutto quando si cerchi di metterla sotto una forma relativisticamente invariante: ad alcune di queste difficoltà accenneremo nel paragrafo seguente (§ 2). Qui diremo solo come dette difficoltà hanno suggerito, con I. FRENKEL, l'opportunità di considerare l'elettrone come una particella puntiforme, nonostante che una siffatta ipotesi porti, almeno a prima vista, ad introdurre nella teoria grandezze infinite e quindi prive di significato.

Un tentativo rivolto a superare queste nuove difficoltà, senza sostanzialmente alterare la concezione maxwelliana del campo elettromagnetico, è stato fatto in tempi recenti da DIRAC e sviluppato ulteriormente anche da altri (ad esempio, ELIEZER) (§ 2). La teoria dell'elettrone classico di Dirac riesce a superare le difficoltà delle grandezze infinite, ma ciò nonostante essa pure non è completamente soddisfacente. Si pensi infatti che l'equazione del moto dell'elettrone contiene le derivate temporali del terzo ordine delle coordinate e conseguentemente tale moto non risulta completamente determinato dai valori iniziali della velocità e della posizione. Non solo, ma tale equazione in alcuni problemi porta, qualunque sia la scelta delle costanti di integrazione, a moti in completa contraddizione con quanto sembra ovvio attendersi in base all'ordinaria intuizione.

Per superare anche queste difficoltà ed arrivare alla formulazione di una teoria classica dell'elettrone più soddisfacente, siamo partiti dalla considerazione che già per superare altre difficoltà della fisica odierna delle particelle elementari, diversi fisici teorici (HEISENBERG, MARCH, LANDÉ, SNYDER, FLINT, BORN, ecc.) hanno affermato la necessità di introdurre nella teoria una nuova costante universale, indipendente da quelle attualmente considerate (h, c), delle dimensioni di una lunghezza e avente l'ordine di grandezza, del cosiddetto raggio classico dell'elettrone (§ 3).

Si arriva così a dedurre per il moto dell'elettrone un'equazione, relativisticamente invariante, che anzichè essere differenziale risulta alle differenze finite (§ 4).

Lo studio di tale equazione e la sua integrazione in diversi problemi particolari, mostra come essa sia esente dalle difficoltà che caratterizzavano le teorie precedenti (§ 5).

Inoltre la dinamica dell'elettrone, quale risulta dalla nostra teoria, può, quando si tratti di moti conservativi, essere sviluppata seguendo il formalismo lagrangiano, pur di generalizzare opportunamente i metodi di OSTROGRADSKI e di PAIS-UHLENBECK: con ciò si ottengono anche interessanti legami coi cosiddetti modelli d'elettrone quale particella polo-dipolare (HÖLN, LUBANSKI, MATHISON, WEYSSENHOFF, ecc.) o con modelli che di questi costituiscono opportune generalizzazioni (BOPP) (§ 6).

Dal punto di vista elettrodinamico è interessante osservare come l'equazione per il moto dell'elettrone da noi stabilita si riduca, nell'approssimazione non relativistica, a un'equazione alle differenze finite che corrisponde a un particolare modello di elettrone esteso già studiato da PAGE, da BOHM e WEINSTEIN e da ELIEZER (§ 7).

Inoltre, e questo ci sembra il risultato più interessante, si è trovato un legame tra la nostra equazione del moto e la teoria maxwelliana del campo elettromagnetico, nel senso che si è mostrato, seguendo un formalismo dovuto a RZEWUSKI, che tale equazione può essere dedotta dalle equazioni del campo pur di ricorrere a un modello relativisticamente invariante di sorgente estesa del tipo di quelli considerati nelle odierne teorie di campi non locali (§ 8).

Come ultimo risultato si è osservato che l'equazione alle differenze finite dell'elettrone, ammette, oltre alle soluzioni che descrivono i moti «esterni» determinati dalla forza agente sull'elettrone, anche soluzioni periodiche che descrivono moti «interni» dell'elettrone: a quest'ultime può essere ricondotta l'esistenza del momento magnetico anomalo dell'elettrone (§ 9).

Concludendo, la teoria sviluppata in questo lavoro sembra portare ad un'equazione soddisfacente per il moto classico dell'elettrone: essa può pertanto costituire un buon punto di partenza per una corrispondente teoria quantistica, la quale potrebbe portare fra l'altro, come conseguenza dell'esistenza dei moti «interni» dell'elettrone, a uno spettro di valori per le masse osservabili (§ 10).

2. – Difficoltà delle odierne teorie classiche dell'elettrone.

Nella sua teoria dell'elettrone sferico contrattile per effetto del suo moto, LORENTZ [1], calcolando la forza di reazione partendo dai potenziali ritardati di Lienard e Wiechert, arriva alla seguente equazione del moto:

$$m_0 \frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} + \boldsymbol{R},$$

[1] H. A. LORENTZ: *The Theory of Electrons* (Leipzig, 1916).

dove

$$m_0 = \frac{2}{3}\frac{e^2}{R},$$

essendo R il raggio a riposo dell'elettrone e denotando

$$\boldsymbol{R} = \frac{2}{3}\frac{e^2}{c^3}\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}t^2} + (\ )R + (\ )R^2 + \ldots$$

la forza di reazione.

Si osservi esplicitamente che i coefficienti delle varie potenze di R contengono le derivate successive della velocità $\boldsymbol{v}$ e che inoltre la loro effettiva espressione cambierebbe se, discostandosi da LORENTZ, si assumesse un modello diverso da quello di una sfera piena a riposo. È però essenziale il fatto che i termini di $\boldsymbol{R}$ successivi al primo si annullerebbero tutti qualora si supponesse, con un passaggio al limite, l'elettrone puntiforme; in questo caso si avrebbe però il risultato assurdo di una massa a riposo m_0 infinita.

Dal punto di vista della teoria del campo elettromagnetico osserviamo ancora che, partendo dal tensore energetico $T_{\alpha\beta}$ del campo elettromagnetico, si possono ricavare la quantità di moto e l'energia associate all'elettrone in moto. Si trovano le espressioni:

$$\boldsymbol{G} = \frac{1}{c}\int_V \boldsymbol{T}_{4\nu}\,\mathrm{d}V = \frac{4}{3}\frac{\boldsymbol{v}}{\sqrt{1-\beta^2}}\frac{U_0}{c} = m_0\frac{\boldsymbol{v}}{\sqrt{1-\beta^2}},$$

$$U = \int_V T_{44}\,\mathrm{d}V = \frac{U_0}{\sqrt{1-\beta^2}}(1+\beta^2/3),$$

che presentano due difficoltà assai serie. La prima consiste nel fatto che, contrariamente a quanto richiesto dal principio relativistico di equivalenza tra massa e energia, sussisterebbe la relazione $m_0 = \frac{4}{3}(U_0/c^2)$; la seconda difficoltà consiste invece nel fatto che le espressioni di $\boldsymbol{G}$, U_0/c non si trasformano come un tetravettore, in evidente contrasto col modo di comportarsi della quantità di moto e dell'energia di una particella.

Per ovviare alle difficoltà della teoria, DIRAC [2], sempre partendo dalle equazioni di Maxwell, calcola il flusso dell'energia-impulso attraverso un tubo di raggio ε assai piccolo ($\varepsilon \ll R$) che circonda la linea d'universo dell'elet-

[2] P. A. M. DIRAC: *Proc. Roy. Soc.*, A **167**, 148 (1938); *Ann. Inst. Poincaré*, **9**, 13 (1938).

trone, arrivando in definitiva alla seguente equazione:

$$m_0 c \frac{\mathrm{d}u_\alpha}{\mathrm{d}s} = F_\alpha + R_\alpha \qquad (u_\alpha u_\alpha = -c^2)\,, \tag{1}$$

dove la massa a riposo m_0 è ottenuta come differenza

$$m_0 = \frac{1}{2}\frac{e^2}{c^2}\frac{1}{\varepsilon} - k(\varepsilon)$$

di due termini che diventano infiniti per $\varepsilon \to 0$, e dove

$$F_\alpha = \frac{e}{c} F_{\alpha\beta} u_\beta\,, \qquad R_\alpha = \frac{2}{3}\frac{e^2}{c}\left(\frac{\mathrm{d}^2 u_\alpha}{\mathrm{d}s^2} + \frac{u_\alpha u_\beta}{c^2}\frac{\mathrm{d}^2 u_\beta}{\mathrm{d}s^2}\right)$$

sono rispettivamente i quadrivettori della forza esterna e della reazione. La (1) nell'approssimazione non relativistica si riduce alla

$$m_0 \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \frac{2}{3}\frac{e^2}{c^3}\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}t^2} = \boldsymbol{F}, \tag{2}$$

da cui si vede che la reazione è uguale al termine $\frac{2}{3}(e^2/c^3)(\mathrm{d}^2\boldsymbol{v}/\mathrm{d}t^2)$ che nella teoria di Lorentz caratterizza l'elettrone puntiforme.

La difficoltà maggiore che si incontra nell'adozione di equazioni del tipo della (1) o della (2) è legata al fatto che, comparendo in esse derivate temporali della velocità di ordine superiore al primo, i loro integrali generali contengono un numero di costanti arbitrarie superiore a due e, attribuendo a queste costanti valori generici, si hanno soluzioni descriventi moti in netto contrasto con le previsioni dell'ordinaria intuizione fisica. Per ovviare a questo inconveniente, diversi autori [3] hanno suggerito opportuni criteri tendenti a discriminare, tra le soluzioni matematiche delle equazioni, le soluzioni « fisiche » da quelle « non fisiche ». Nessuno di tali criteri ha però un carattere completamente generale. Per di più in alcuni problemi particolari come quello, sia pure un po' patologico, di un elettrone lanciato lungo una retta contro un protone « fisso », si hanno [4] come soluzioni delle (1) e delle (2) risultati che, indipendentemente dai valori assunti dalle costanti arbitrarie di integrazione, non sembrano in alcun modo accettabili.

[3] M. Schönberg: *Phys. Rev.*, **76**, 211 (1945); C. J. Eliezer e A. W. Mailvaganam: *Proc. Camb. Phil. Soc.*, **41**, 148 (1945); P. Caldirola: *Nuovo Cimento*, **5**, 99 (1948); A. Loinger: *Nuovo Cimento*, **6**, 360 (1949).

[4] C. J. Eliezer: *Proc. Camb. Phil. Soc.*, **33**, 173 (1943).

Modifiche della teoria di Dirac sono state proposte da ELIEZER [5]. Esse si basano sull'osservazione che, contrariamente a quanto fatto da LORENTZ e da DIRAC, si possono assumere per il campo generato dall'elettrone in moto espressioni più generali di quelle derivanti dai potenziali ritardati.

Ed infatti prendendo per il tensore $F_{\alpha\beta}$ l'espressione:

$$F_{\alpha\beta} = k(F_{\alpha\beta}^{(\mathrm{rit})} - F_{\alpha\beta}^{(\mathrm{ant})}) + F_{\alpha\beta}^{(\mathrm{rit})}$$

(con k costante) si arriva, anzichè alla (1), alla seguente espressione per la forza di reazione che agisce sull'elettrone:

$$R_\alpha = \frac{2}{3}\frac{e^2}{c}(2k+1)\left(\frac{\mathrm{d}^2 u_\alpha}{\mathrm{d}s^2} + \frac{u_\alpha u_\beta}{c^2}\frac{\mathrm{d}^2 u_\beta}{\mathrm{d}s^2}\right).$$

In una serie di applicazioni ELIEZER mostra come, assumendo $2k+1<0$, si possano ottenere risultati soddisfacenti anche nel problema «critico» del moto relativo di due cariche: esistono però purtroppo, come ha mostrato ZIN [6], altri casi caratterizzati dal fatto che le soluzioni delle equazioni del moto dell'elettrone sono essenzialmente di tipo non fisico.

Le difficoltà della teoria dell'elettrone classico di Dirac rimangono quindi inalterate anche nella teoria generalizzata di Eliezer.

3. – Introduzione di una lunghezza fondamentale.

In diversi campi dell'odierna fisica delle particelle elementari si è riconosciuta, in questi ultimi anni, la necessità di introdurre, accanto alla costante h di Planck e alla velocità c della luce, una nuova costante universale indipendente, ad esempio delle dimensioni di una lunghezza.

Tale costante si presenta necessaria allo scopo di rendere convergenti alcune espressioni che, nell'odierna teoria dei campi associati alle particelle elementari, si presentano sotto forma di integrali divergenti.

HEISENBERG [7] è stato il primo a prospettare seriamente la introduzione nell'odierna fisica teorica di una lunghezza fondamentale λ invariante la quale, combinata con le altre costanti h e c, permetterebbe anche di esprimere ogni grandezza fisica in funzione delle tre predette costanti. Egli, basandosi sullo studio di alcuni processi relativi alle particelle elementari, afferma inoltre che la costante λ deve avere lo stesso ordine di grandezza del raggio classico dell'elettrone

$$\lambda \cong r_0 = \frac{e^2}{m_0 c^2} = 2{,}81 \cdot 10^{-13}\ \mathrm{cm}.$$

[5] C. J. ELIEZER: *Rev. Mod. Phys.*, **19**, 147 (1947).
[6] G. ZIN: *Nuovo Cimento*, **6**, 1 (1949).
[7] W. HEISENBERG: *Ann. der Phys.*, **32**, 20 (1938).

Un analogo punto di vista è adottato anche da MARCH [8], secondo il quale l'introduzione della nuova costante universale è richiesta dal fatto che è concettualmente impossibile, con una esperienza di qualsiasi specie, distinguere le posizioni di due particelle in quiete la cui distanza sia inferiore a un certo limite λ; il che, in altri termini, equivale ad affermare l'impossibilità di localizzare una particella in quiete con una precisione superiore a λ.

Altri punti di vista sono invece quelli seguiti da LANDÉ [9] e da BORN [10] che si fondano sull'introduzione, accanto alla relazione classica di Einstein:

$$\left(\frac{E}{c}\right)^2 - p^2 = m_0^2 c^2 ,$$

di un'altra relazione (detta «equazione del segnale»):

$$(c\,\Delta t)^2 - (\Delta r)^2 = \lambda^2 ,$$

che pone un limite inferiore λ alle distanze che si considerano.

Considerazioni più drastiche sono infine quelle sviluppate da SNYDER [11] e da FLINT [12] che introducono una lunghezza fondamentale come conseguenza di una intrinseca quantizzazione dello spazio-tempo. Secondo SNYDER le coordinate x, y, z, t vanno considerate alla stregua di osservabili rappresentabili da operatori fra loro non commutabili; nelle regole di commutazione cui esse soddisfano compare la nuova costante λ.

Ricordiamo ancora come un'altra via seguita per l'introduzione di una lunghezza fondamentale nelle equazioni della fisica moderna ci è offerta dalle odierne teorie dei cosiddetti «campi non locali» (WATAGHIN, YUKAWA, MOLLER, RAYSKI, ecc.). Esse si possono sostanzialmente ricondurre all'introduzione nella lagrangiana di interazione, al posto dell'ordinario termine $j(x)A(x)$, di una espressione del tipo

$$\int\!\!\int F(x-x';\, x-x'')\, j(x')A(x'')\, \mathrm{d}x'\, \mathrm{d}x''$$

che definisce appunto quella che si suol chiamare una «interazione non locale». La funzione $F(x-x';\, x-x'')$, detta *fattore di forma*, essendo adimensionale dovrà necessariamente contenere la lunghezza fondamentale λ; evidenti ragioni fisiche impongono che la F sia sensibilmente diversa da zero solo quando $|x-x'|$ e $|x-x''$ sono inferiori o dell'ordine di λ. Il formalismo dei campi

(8) A. MARCH: *Quantum Mechanics of Particles and Wave Fields* (New York, 1951).
(9) A. LANDÉ: *Journ. Franklin Inst.*, **229**, 767 (1940).
(10) M. BORN: *Rev. Mod. Phys.*, **21**, 463 (1949).
(11) H. S. SNYDER: *Phys. Rev.*, **71**, 38 (1947).
(12) H. T. FLINT: *Phil. Mag.*, **7**, 29, 33 (1940); *Phys. Rev.*, **74**, 209 (1948).

ad interazione non locale permette di superare alcune tipiche difficoltà associate all'assunzione del carattere puntiforme delle particelle senza attribuire a queste in modo esplicito una forma e un'estensione finita, il che solleverebbe grosse difficoltà dal punto di vista dell'invarianza relativistica e del processo di quantizzazione. Dal punto di vista matematico l'introduzione della lunghezza elementare λ, attraverso l'ipotesi di un'interazione non locale, comporta nello studio dei campi in interazione la sostituzione delle equazioni differenziali con equazioni integrali; a questo fatto sono connesse alcune difficoltà che si incontrano quando si vogliono esprimere le leggi di conservazione (della carica-corrente, dell'energia-impulso) per mezzo di equazioni del tipo di quelle cosiddette di continuità.

4. – Equazione relativistica alle differenze finite per la dinamica dell'elettrone.

Ci proponiamo ora di vedere se le difficoltà che si incontrano nella teoria odierna nello stabilire l'equazione fondamentale della dinamica dell'elettrone irraggiante possano essere superate introducendo in tale equazione la lunghezza fondamentale di cui si è discorso al paragrafo precedente. In questo tentativo cercheremo dapprima di stabilire la nuova equazione per la dinamica dell'elettrone direttamente, vale a dire senza preoccuparci di chiarire il suo legame con le equazioni del campo elettromagnetico; su tale legame verrà invece indagato in seguito.

A questo punto vogliamo ricordare come già qualche autore abbia tentato di sostituire alle ordinarie equazioni differenziali della fisica delle equazioni alle differenze finite. Fra i vari tentativi ricorderemo quello di DARLING ([13]), la cui idea fondamentale è che « le particelle elementari non hanno un'esistenza individuale a sè, ma sono invece la manifestazione attuale di un processo materiale i cui aspetti sono la creazione, l'esistenza continuata, l'annichilazione di particelle »: questo processo richiede l'esistenza di un « volume » elementare non nullo dello spazio-tempo. Per questa via DARLING arriva a scrivere le equazioni ondulatorie dei campi (di Maxwell, di Dirac, di Proca, ecc.) sotto forma di equazioni alle differenze finite. La teoria di Darling, anche negli sviluppi successivi dovuti a DARLING e ZILSEL ([14]), si presenta in qualche punto assai artificiosa e non esente da critiche, onde allo stato attuale è difficile dire quale sia l'effettivo progresso da essa portato nel tentativo di superare le odierne difficoltà della fisica delle particelle elementari.

In vista della complessità dell'argomento, noi riteniamo più conveniente circoscrivere il problema da affrontare; ci limiteremo cioè, almeno in un primo tempo, a considerare il problema del moto classico dell'elettrone irraggiante.

([13]) B. T. DARLING: *Phys. Rev.*, **80**, 460 (1950).
([14]) B. T. DARLING e P. R. ZILSEL: *Phys. Rev.*, **91**, 1252 (1953).

In relazione a questo programma ci siamo proposti ([15]) di stabilire un'equazione per la dinamica dell'elettrone prendendo come punto di partenza la seguente serie di postulati:

1) esiste una costante universale τ_0 dalle dimensioni di un tempo (intervallo di tempo fondamentale);

2) una forza, agente su di una particella all'istante τ del tempo proprio di questa, determina un brusco passaggio dallo stato di moto relativo all'istante $\tau - \tau_0$ allo stato di moto relativo all'istante τ. In altri termini le leggi della meccanica connettono grandezze dinamiche della particella relative a punti della linea oraria cronotopica distanti l'uno dall'altro di una lunghezza elementare $s_0 = c\tau_0$;

3) per $\tau_0 \to 0$ le nuove equazioni si devono ridurre a quelle della dinamica classica;

4) le leggi devono essere invarianti relativisticamente.

Vogliamo notare esplicitamente come l'introduzione di un intervallo di tempo elementare, e quindi di una lunghezza fondamentale, secondo i postulati precedenti non comporti alcuna quantizzazione, nè alcuna concezione di struttura reticolare dello spazio tempo. Conseguentemente avrà senso considerare le coordinate x_α come funzioni continue di un parametro qualsiasi, ad esempio, l'arco di linea s e considerare, punto per punto, le derivate $\mathrm{d}x_\alpha/\mathrm{d}s$.

Vogliamo ora vedere come i postulati assunti ci permettono di stabilire per la dinamica dell'elettrone un'equazione alle differenze finite. Allo scopo determiniamo dapprima, per semplicità, le leggi del moto dell'elettrone quando provvisoriamente si lasci cadere la condizione 4). Detta allora $\boldsymbol{F}(\boldsymbol{r}; t)$ la forza che agisce sulla particella all'istante t, essa produrrà una variazione sul moto di questa in modo da farle acquistare una quantità di moto $m_0\boldsymbol{v}(t)$. Tenendo conto dei postulati 1), 2), 3) assumeremo a base della nostra teoria la seguente equazione alle differenze finite:

$$\frac{m_0}{\tau_0}[\boldsymbol{v}(t) - \boldsymbol{v}(t - \tau_0)] = \boldsymbol{F}(\boldsymbol{r}; \boldsymbol{v}; t) \tag{3}$$

ove, conformemente a un'osservazione precedente, si deve intendere

$$\boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t} .$$

([15]) cfr. P. Caldirola: *Nuovo Cimento*, **10**, 1747 (1953).

Sviluppiamo $\boldsymbol{v}(t-\tau_0)$ in una serie di potenze crescenti del parametro τ_0:

$$\boldsymbol{v}(t-\tau_0) = \boldsymbol{v}(t) - \tau_0 \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{\tau_0^2}{2} \frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}t^2} - \dots + \dots .$$

In approssimazione zero avremo:

$$m_0 \frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{r};\, \boldsymbol{v};\, t) ,$$

che evidentemente è l'equazione del moto di una particella sottoposta alla forza esterna $\boldsymbol{F}$ e senza reazione.

In prima approssimazione avremo invece:

$$m_0 \left[\frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t} - \frac{\tau_0}{2} \frac{\mathrm{d}^2\boldsymbol{v}(t)}{\mathrm{d}t^2} \right] = \boldsymbol{F}(\boldsymbol{r};\, \boldsymbol{v};\, t) ,$$

che si riduce all'equazione non relativistica (2) di Dirac-Lorentz quando si assuma

$$\tau_0 = \frac{4}{3} \frac{e^2}{m_0 c^3} .$$

In questa approssimazione, posto $R = \tau_0 c/2$, avremo per la massa dell'elettrone $m_0 = \frac{2}{3}(e^2/Rc^2)$.

Si osservi infine come, considerando tutti i termini dello sviluppo di $\boldsymbol{v}(t-\tau_0)$, la (3) equivalga ad un'equazione differenziale in cui compaiano derivate di ordine qualsiasi: ciò nonostante, come vedremo, il problema delle condizioni iniziali non dà luogo a difficoltà.

Procederemo ora alla generalizzazione dell'equazione (3) in modo da renderla relativisticamente invariante. Tenendo conto che $F_\alpha = (e/c) F_{\alpha\beta} u_\beta$, avremo che il primo membro della (3) dovrà essere sostituito da una nuova espressione X_α che dovrà soddisfare alle seguenti condizioni:

a) per $\beta^2 \to 0$ le prime tre componenti di X_α si dovranno ridurre a $(m_0/\tau_0)[\boldsymbol{v}(t) - \boldsymbol{v}(t-\tau_0)]$;

b) per $\tau_0 \to 0$ si dovrà avere

$$X_\alpha \to m_0 c \frac{\mathrm{d}u_\alpha}{\mathrm{d}s} - \frac{2}{3} \frac{e^2}{c} \left[\frac{\mathrm{d}^2 u_\alpha}{\mathrm{d}s^2} + \frac{u_\alpha u_\beta}{c^2} \frac{\mathrm{d}^2 u_\beta}{\mathrm{d}s^2} \right] ;$$

c) essendo $F_{\alpha\beta} u_\alpha u_\beta = 0$, dovrà essere $X_\alpha u_\alpha = 0$.

Queste condizioni determinano la X_α che risulta:

$$X_\alpha = \frac{m_0}{\tau_0}\left\{u_\alpha(\tau) - u_\alpha(\tau-\tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}[u_\beta(\tau) - u_\beta(\tau-\tau_0)]\right\} =$$

$$= -\frac{m_0}{\tau_0}[u_\alpha(\tau-\tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}u_\beta(\tau-\tau_0)]\,,$$

essendo $\tau = s/c$ il tempo proprio della particella.

L'equazione relativistica alle differenze finite per il moto dell'elettrone si scrive quindi:

$$(4) \qquad -\frac{m_0}{\tau_0}[u_\alpha(\tau-\tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}u_\beta(\tau-\tau_0)] = \frac{e}{c}F_{\alpha\beta}(x_\alpha;\, u_\alpha;\, \tau)u_\beta(\tau)\,,$$

essendo

$$u_\alpha(\tau) = \frac{\mathrm{d}x_\alpha(\tau)}{\mathrm{d}\tau}\,.$$

Si noti che nello stabilire questa equazione si è ammesso (postulato 3) che le azioni agenti sull'elettrone avessero un carattere « ritardato ». L'equazione analoga alla (4), qualora si considerassero azioni a carattere « anticipato » sarebbe la seguente:

$$(5) \qquad \frac{m_0}{\tau_0}[u_\alpha(\tau+\tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}u_\beta(\tau+\tau_0)] = \frac{e}{c}F_{\alpha\beta}u_\alpha(\tau)\,,$$

cui corrisponde nell'approssimazione non relativistica:

$$(6) \qquad \frac{m_0}{\tau_0}[\boldsymbol{v}(t+\tau_0) - \boldsymbol{v}(t)] = \boldsymbol{F}\,.$$

Supponendo invece che le azioni siano date dalla semisomma delle azioni ritardate, e di quelle anticipate, si arriva all'equazione simmetrica:

$$(7) \qquad \frac{m_0}{2\tau_0}\left\{u_\alpha(\tau+\tau_0) - u_\alpha(\tau-\tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}[u_\beta(\tau+\tau_0) - u_\beta(\tau-\tau_0)]\right\} = \frac{e}{c}F_{\alpha\beta}u_\beta(\tau),$$

cui corrisponde nell'approssimazione non relativistica:

$$(8) \qquad \frac{m_0}{2\tau_0}[\boldsymbol{v}(t+\tau_0) - \boldsymbol{v}(t-\tau_0)] = \boldsymbol{F}\,.$$

Come vedremo meglio in seguito l'equazione fisicamente accettabile è la (4), che contiene le perdite per irraggiamento subite dall'elettrone. La (7) vale

invece quando si trascurano queste perdite: il moto da essa descritto è conservativo ([16]).

Dal punto di vista matematico il moto dell'elettrone è dunque descritto da equazioni inomogenee, ad esempio le (4), alle differenze finite nelle derivate $u_\alpha = \mathrm{d}x_\alpha/\mathrm{d}\tau$ delle incognite x_α.

Orbene si può dimostrare ([17]) che l'integrale generale contiene due costanti arbitrarie oltre eventualmente delle funzioni periodiche arbitrarie soluzioni dell'equazione

$$u_\alpha(\tau) - u_\alpha(\tau - \tau_0) = 0 .$$

Dato però che il moto della particella lungo la sua traiettoria in virtù del postulato 2), risulta definito, quando si assuma come istante iniziale $\tau = 0$, solo negli istanti $\tau = n\tau_0$ (con n intero), la considerazione di tali soluzioni periodiche può essere esclusa. Il moto così considerato verrà da noi denominato « macroscopico ».

Ci si potrà a suo tempo porre la domanda se alle soluzioni periodiche non corrispondano dei moti effettivi dell'elettrone, moti che chiameremo « interni »: questi, in quanto concernono il comportamento dell'elettrone a distanze inferiori a circa $R \cong 2\tau_0/c$, potranno essere connessi a problemi di « struttura » dell'elettrone stesso.

Per ora, possiamo concludere che la legge del moto macroscopico (vale a dire il moto che si ottiene prescindendo dalle soluzioni interne) dell'elettrone, che si ricava integrando la (4), è completamente determinata assegnando il valore di due costanti (vettoriali) a priori arbitrarie: questa proprietà corregge pertanto il difetto più grave delle teorie di Lorentz e di Dirac.

Si noti infine che la comparsa di derivate superiori nelle teorie precedenti di Lorentz e di Dirac si comprende osservando, come già si è detto, che sviluppando le funzioni $u_\alpha(\tau - \tau_0)$ in serie di potenze di τ_0 l'equazione alle differenze finite si trasforma in un'equazione differenziale di ordine infinito.

Terminiamo con l'osservare come all'equazione non relativistica (3) fossero giunti già da tempo Eliezer e Bohm e Weinstein in base a considerazioni che verranno illustrate in seguito (cfr. § 7). Inoltre a un'equazione

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([16]) Si noti che l'equazione alle azioni anticipate (5) diventa equivalente a quella alle azioni ritardate (4) quando in essa si metta come campo $F_{\alpha\beta}$ quello $F^{(\mathrm{ant})}_{\alpha\beta}$ derivante dai potenziali elettromagnetici anticipati. Ne segue allora l'equivalenza con la (5) e la (4) anche dell'equazione (7) in cui però per $F_{\alpha\beta}$ si intenda la semisomma fra $F^{(\mathrm{ant})}_{\alpha\beta}$ e $F^{(\mathrm{rit})}_{\alpha\beta}$. Per la dimostrazione di queste osservazioni, nel caso limite di un elettrone puntiforme, si veda: J. A. Wheeler e R. P. Feynman: *Rev. Mod. Phys.*, **17**, 157 (1945).

([17]) S. veda ad esempio: E. Pascal: *Calcolo delle variazioni e delle differenze finite* (Milano, 1897).

alle differenze finite non relativistica diversa dalla (3) e precisamente alla

$$\frac{m_0}{\tau_0^2}[\boldsymbol{r}(t)-2\boldsymbol{r}(t-\tau_0)+\boldsymbol{r}(t-2\tau_0)]=\boldsymbol{F}$$

sono pervenuti MÖGLICH e ROMPE [18] partendo da un'ipotesi di un tempo « essenzialmente » discontinuo e quantizzato. In altri termini questi autori fanno l'ipotesi che la derivata temporale di una funzione qualsiasi $f(t)$ del tempo vada *fisicamente* sostituita dal rapporto $[f(t)-f(t-\tau_0)]/\tau_0$.

5. – Applicazioni varie.

L'integrazione dell'equazione relativistica alle differenze finite (3), o anche di quella corrispondente non relativistica, è in generale molto complicata. Ci limiteremo pertanto a trattare qualche caso particolare.

5·1. *Particella libera.* – L'equazione relativistica, ponendo $F_{\alpha\beta}=0$, diventa

$$u_\alpha(\tau-\tau_0)=-\frac{u_\alpha(\tau)u^\beta(\tau)}{c^2}u_\beta(\tau-\tau_0),$$

che ha come unica soluzione macroscopica:

$$u_\alpha(n\tau_0)=u_\alpha(\tau_0)=\text{cost}\,.$$

Il moto è cioè rettilineo e uniforme.

5·2. *Elettrone sottoposto a campo elettrico costante E.* – Le equazioni relativistiche, assumendo la direzione del campo come asse x e supponendo che per $t=0$ si abbia $u_1=u_2=u_3=0$, si scrivono:

$$-\frac{m_0}{\tau_0}u_1(\tau-\tau_0)+\frac{u_1(\tau)}{c^2}[u_1(\tau)u_1(\tau-\tau_0)-u_4(\tau)u_4(\tau-\tau_0)]=eEu_4(\tau)\,,$$

$$-\frac{m_0}{\tau_0}u_4(\tau-\tau_0)+\frac{u_4(\tau)}{c^2}[u_1(\tau)u_1(\tau-\tau_0)-u_4(\tau)u_4(\tau-\tau_0)]=eEu_1(\tau)\,.$$

Ponendo

$$\chi=\frac{\tau_0}{m_0}\frac{eE}{c^2}$$

e applicando un procedimento di iterazione si arriva alla seguente espressione

[18] L. MÖGLICH e R. ROMPE: *Zeits. f. Phys.*, **113**, 740 (1939).

esplicita per $u_1(n\tau_0)$:

$$\frac{u_1(n\tau_0)}{c^2} = n\chi(1+\chi^2)^{(n-1)/2} + \sum_2^{[(n+2)/2]}{}_k (n-2k+2)\chi^{2k-1}(1+\chi^2)^{(n-2k+1)/2}$$

(l'estremo superiore della sommatoria $[(n+2)/2]$ vuol indicare il massimo intero che non supera $(n+2)/2$).

Nella approssimazione non relativistica la formula precedente si riduce a

$$\boldsymbol{v}(n\tau_0) = n\chi c^2 = \frac{eE}{m_0} n\tau_0 ,$$

che mostra come il moto sia uniformemente accelerato.

5·3. *Particella sottoposta a una forza dipendente dal tempo* $\boldsymbol{F} = \boldsymbol{F}(t)$. – L'equazione non relativistica

$$\frac{m_0}{\tau_0}[\boldsymbol{v}(t) - \boldsymbol{v}(t-\tau_0)] = \boldsymbol{F}(t)$$

ha come soluzione:

$$\boldsymbol{v}(t) = \boldsymbol{v}_0 + \frac{\tau_0}{m_0} \sum_1^n{}_r \boldsymbol{F}(r\tau_0).$$

In particolare:

a) se

$$\boldsymbol{F}(t) = \boldsymbol{K} \exp[-\alpha t],$$

si ha

$$\boldsymbol{v}(n\tau_0) = \boldsymbol{v}_0 + \frac{\tau_0}{m_0} \boldsymbol{K} \frac{1 - \exp(-\alpha n\tau_0)}{\exp(\alpha\tau_0) - 1} ;$$

b) se

$$\boldsymbol{F}(t) = e\boldsymbol{E}_0 \sin \omega t ,$$

si ha

$$\boldsymbol{v}(n\tau_0) = \boldsymbol{v}_0 + \frac{\tau_0}{m_0} e\boldsymbol{E}_0 \frac{\cos(\omega\tau_0/2) - \cos[(n+\frac{1}{2})\omega\tau_0]}{2\sin(\omega\tau_0/2)} ;$$

c) se

$$\boldsymbol{F}(t) = \boldsymbol{F}_0\, \delta(t - t_0) \qquad \text{(impulso istantaneo)},$$

si ha (posto $t_0 = r\tau_0$ e assunto $\boldsymbol{v}(0) = 0$):

$$\boldsymbol{v}(n\tau_0) = 0 \qquad \text{per } n < r ,$$

$$\boldsymbol{v}(n\tau_0) = \frac{\tau_0}{m_0} \boldsymbol{F}_0 \qquad \text{per } n \geqslant r ,$$

cioè la particella, ricevuto l'impulso, si muove di moto uniforme.

Notiamo che trattando questo problema con la corrispondente equazione di Dirac-Lorentz si ha:

$$\boldsymbol{v}(t) = \frac{2}{3}\frac{e^2}{m_0 c^3}\boldsymbol{F}_0 \exp\left[\frac{3}{2}\frac{m_0 c^3}{e^2}(t - t_0)\right] \qquad (\text{per } t < t_0)$$

$$\boldsymbol{v}(t) = \frac{2}{3}\frac{e^2}{m_0 c^3}\boldsymbol{F}_0 \qquad (\text{per } t \geqslant t_0)$$

che corrisponde a una particella che acquista gradualmente un'accelerazione fino a che viene raggiunta dall'impulso. Questa soluzione è in contrasto, a meno di ricorrere a modelli più complessi di elettrone, col principio di causalità.

5·4. *Elettrone in un campo magnetico costante*: $\boldsymbol{F} = e\boldsymbol{v} \wedge \boldsymbol{H}_0$. – L'equazione

$$\frac{m_0}{\tau_0}[\boldsymbol{v}(t) - \boldsymbol{v}(t - \tau_0)] = e\boldsymbol{v} \wedge \boldsymbol{H}_0$$

ammette la soluzione

$$\boldsymbol{v} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}$$

con

$$\dot{x}(n\tau_0) = \quad v_0(1 + \tau_0^2 e^2 H_0^2/m_0^2)^{-n/2}\cos[n \operatorname{arctg}(\tau_0 e H_0/m_0)],$$

$$\dot{y}(n\tau_0) = -v_0(1 + \tau_0^2 e^2 H_0^2/m_0^2)^{-n/2}\sin[n \operatorname{arctg}(\tau_0 e H_0/m_0)],$$

che corrisponde a una particella che descrive una spirale decrescente.

5·5. *Particella sottoposta a una forza elastica di richiamo* $F = -kx$. – L'equazione

$$\frac{m_0}{\tau_0}[\dot{x}(t) - \dot{x}(t - \tau_0)] = -kx$$

ha per soluzione

$$x(n\tau_0) = A \exp[\alpha n \tau_0],$$

essendo α una costante complessa determinata dall'equazione trascendente

$$\alpha(1 - \exp[-\alpha\tau_0]) = -k\tau_0/m_0 ;$$

essendo la parte reale di α necessariamente negativa, il moto risulta oscillatorio smorzato.

5·6. *Elettrone lanciato lungo una retta contro un protone fisso*:

$$F = -\frac{e^2}{x^2}\,\varepsilon(x) \qquad \text{con} \qquad \varepsilon(x) = \begin{cases} +1 & \text{se } x > 0 \\ -1 & \text{se } x < 0 \end{cases}.$$

Pur non essendo riusciti a risolvere esplicitamente l'equazione

$$\frac{m_0}{\tau_0}[\dot{x}(t) - \dot{x}(t - \tau_0)] = -\frac{e^2}{x^2}\,\varepsilon(x)\,,$$

crediamo tuttavia di poter affermare, in base a una discussione qualitativa di questa equazione, che il moto risulta oscillatorio smorzato e che conseguentemente non si hanno le soluzioni non fisiche che caratterizzano questo problema nella teoria dell'elettrone classico di Dirac.

In modo analogo si possono risolvere i corrispondenti problemi partendo dall'equazione simmetrica (7) o dalla sua approssimazione non relativistica (8). Si trova fra l'altro che, sotto l'azione di forze conservative, i moti sono stazionari come nelle corrispondenti equazioni della dinamica di Einstein e di Newton: le frequenze dei moti periodici, per effetto del campo di reazione, risultano tuttavia variate.

6. – Formulazione lagrangiana della dinamica dell'elettrone.

La formulazione lagrangiana e hamiltoniana della teoria è richiesta fra l'altro allo scopo di poter sottoporre le equazioni classiche al processo di quantizzazione.

La possibilità di una siffatta formulazione è però limitata, come è noto, ai sistemi conservativi. È infatti solo per questi che le equazioni del moto si possono dedurre dal principio variazionale:

$$\delta \int_{t_1}^{t_2} L\,dt = 0\,, \tag{9}$$

essendo L la funzione lagrangiana del sistema.

In questo paragrafo ci proponiamo di trovare la lagrangiana corrispondente all'equazione simmetrica (7) (o alla sua approssimazione non relativistica (8)).

Osserviamo anzitutto che tali equazioni possono anche scriversi simboli-

camente sotto la forma:

$$\frac{m_0}{\tau_0}\left\{\sinh\left(\tau_0\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\cdot u_\alpha(\tau)+\frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}\sinh\left(\tau_0\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\cdot u_\beta(\tau)\right\}=\frac{e}{c}F_{\alpha\beta}u_\beta(\tau)$$

e

$$\frac{m_0}{\tau_0}\sinh\left(\tau_0\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\cdot\boldsymbol{v}(t)=\boldsymbol{F}\,,$$

nelle quali appare evidente la comparsa di derivate di ordine qualsiasi nelle coordinate delle particelle.

Ciò induce a tentare una formulazione lagrangiana ed hamiltoniana seguendo due metodi, l'uno dovuto a Ostrogradski ([19]) e l'altro a Pais e Uhlenbeck ([20]), sviluppati allo scopo di trattare sistemi lagrangiani generalizzati in cui la funzione di Lagrange dipende dalle derivate temporali delle coordinate di ordine qualsiasi.

Seguiremo dapprima il metodo di Ostrogradski, incominciando dall'approssimazione non relativistica.

Supposto quindi che la lagrangiana L sia una funzione del tempo t, delle f coordinate libere q_i e delle derivate temporali $q_i^{(n)}$ delle q_i di ordine qualsiasi, cioè

$$L=L(\,q\,|\,\dot{q}\,|\,\ddot{q}\ldots|\,q^{(n)}\,|\ldots;\,t)\,.$$

Eseguendo la variazione indicata in (9), con immediata integrazione per parti, si ottengono le equazioni di moto lagrangiane:

$$\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right)+\frac{\mathrm{d}^2}{\mathrm{d}t_2}\left(\frac{\partial L}{\partial\ddot{q}_i}\right)-\ldots+\ldots=\sum_n(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{\partial L}{\partial q_i^{(n)}}\right)=0\,.$$

D'altra parte sviluppando la (8) in serie di potenze di τ_0, si ha:

$$\frac{m_0}{\tau_0}\left[\dot{\boldsymbol{v}}(t)+\frac{\tau_0^3}{3!}\dddot{\boldsymbol{v}}(t)-\frac{\tau_0^5}{5!}\overset{(5)}{\boldsymbol{v}}(t)+\ldots\right]=\boldsymbol{F}\,.$$

Confrontando questa con la precedente si ottiene immediatamente per la lagrangiana (assunte come coordinate libere x, y, z) l'espressione:

$$L=\frac{m_0}{2\tau_0}\sum_0^\infty(-1)^r\frac{\tau_0^{2r+1}}{(2r+1)!}\overset{(r)}{\boldsymbol{v}}\cdot\overset{(r)}{\boldsymbol{v}}-U\,, \tag{10}$$

avendo indicato con U la funzione potenziale della forza esterna $\boldsymbol{F}$.

([19]) Ostrogradsky: *Mém. de l'Acad. de St. Pétersbourg*, **6**, 385 (1850); cfr. E. T. Whittaker: *A Treatise on the Analitical Dynamics* (Cambridge, 1917), Cap. X, e per generalizzazioni: J. Weyssenhoff: *Acta Phys. Polonica*, **11**, 49 (1951).

([20]) A. Pais e G. E. Uhlenbeck: *Phys. Rev.*, **78**, 145 (1950).

Per ricercare l'hamiltoniana supponiamo dapprima che L dipenda dalle derivate temporali delle coordinate libere fino ad un ordine N finito. Secondo Ostrogradski si possono allora definire le variabili canoniche coniugate:

$$(11) \qquad \begin{cases} Q_{ki} = D^{i-1} q_k \\ P_{ki} = \dfrac{\delta L}{\delta (D^i q_k)} \end{cases} \qquad \begin{pmatrix} i = 1, 2, \ldots, N \\ k = 1, 2, \ldots, f \end{pmatrix}$$

avendo indicato con D l'operatore $\mathrm{d}/\mathrm{d}t$ e con $\delta/\delta(x)$ la derivata variazionale

$$\frac{\delta L}{\delta (x)} = \frac{\partial L}{\partial x} - D \frac{\partial L}{\partial (Dx)} + \ldots .$$

Allora l'hamiltoniana è

$$H = P_{11} Q_{12} + P_{12} Q_{13} + \ldots + P_{1N-1} Q_{1N} + P_{1N} (D^N q_1) + \ldots$$
$$\ldots + P_{kN-1} Q_{kN} + P_{kN} (D^N q_k) - L ,$$

in cui le $D^N q_j$ si eliminano mediante le (11).

Volendo estendere il procedimento per $N \to \infty$, si vede subito che si incontrano difficoltà praticamente insormontabili. Non solo, ma già per un N sufficientemente elevato, il procedimento esposto è assai pesante e quindi di scarsa utilità.

Per convincerci di ciò basta esaminare un lavoro di Waldmann ([21]) in cui viene trattato col metodo esposto, ed allo scopo di procedere alla sua quantizzazione, un'equazione di moto dell'elettrone con derivate fino al 6° ordine.

È interessante vedere, in relazione anche ad alcune teorie dell'elettrone sviluppate da altri Autori ([22]), la generalizzazione relativistica del precedente formalismo di Ostrogradski.

Limitandosi a considerare una particella puntiforme, prenderemo in esame una lagrangiana del tipo:

$$12) \qquad L = - m_0 c^2 - \frac{m_0}{2\tau_0} \sum_1^\infty{}_r (-1)^r \frac{\tau_0^{2r+1}}{(2r+1)!} \overset{(r)}{u}_\alpha \overset{(r)}{u}_\alpha + \frac{e}{c} u_\alpha \Phi_\alpha ,$$

con la condizione supplementare:

$$F = \frac{m_0}{2\tau_0} u_\alpha^2 = - \frac{m_0}{2\tau_0} c^2 .$$

([21]) L. Waldmann: *Zeits. f. Naturf.*, 8*a*, 329 (1953).

([22]) Cfr.: H. Hönl: *Feldmechanik des Elektrons und der Elementarteilchen*, in *Ergebnisse der Exakten Naturwissenschaften*, **26**, 315 e segg. (1952).

Il principio variazionale

$$\delta \int_{\tau_0}^{\tau_1} L(x_\alpha, u_\alpha, \dot{u}_\alpha, \ldots)\, \mathrm{d}\tau = 0$$

porta allora alle seguenti equazioni del moto:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\partial(L+\eta F)}{\partial u_\alpha} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{u}_\alpha} + \frac{\mathrm{d}}{\mathrm{d}\tau^2}\frac{\partial L}{\partial \ddot{u}_\alpha} - \cdots\right] - \frac{\partial L}{\partial x_\alpha} = 0\,,$$

essendo η un moltiplicatore di Lagrange.

Esplicitando le equazioni precedenti otteniamo:

$$\frac{m_0}{2\tau_0}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(2\eta u_\alpha + \frac{2\tau_0^3}{3!}\dddot{u}_\alpha + \ldots\right) = \frac{e}{c}\left(\frac{\partial \Phi_\beta}{\partial x_\alpha} - \frac{\partial \Phi_\alpha}{\partial x_\beta}\right) = \frac{e}{c} F_{\alpha\beta} u_\beta\,.$$

La condizione supplementare:

$$F_{\alpha\beta} u_\beta u_\alpha = 0$$

ci servirà per determinare il moltiplicatore η.

Sviluppando i calcoli si può vedere che quest'ultima condizione è soddisfatta pur di prendere

$$\frac{m_0}{2\tau_0}\frac{\mathrm{d}}{\mathrm{d}\tau}(2\eta u_\alpha) = m_0 \dot{u}_\alpha + \frac{m_0}{2\tau_0}\frac{u_\alpha u_\beta}{c^2}\left[u_\beta(\tau+\tau_0) - u_\beta(\tau-\tau_0)\right],$$

per cui l'equazione del moto diventa proprio la nostra equazione relativistica simmetrica.

Vogliamo notare che se eseguiamo uno sviluppo nelle potenze di τ_0 e ci limitiamo a considerare i termini fino al quarto ordine l'equazione del moto si riduce alla:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left(k_0 - \frac{3}{2} k_1 \dot{u}_\beta^2\right) u_\alpha + k_1 \ddot{u}_\alpha\right\} = \frac{e}{c} F_{\alpha\beta} u_\alpha \tag{13}$$

e la lagrangiana alla

$$L = -k_0 - \frac{1}{2} k_1 \dot{u}_\alpha^2 + \frac{e}{c}\Phi_\alpha u_\alpha\,, \tag{14}$$

in cui i coefficienti k_0 e k_1 hanno i seguenti valori:

$$k_0 = m_0 c^2\,, \qquad k_1 = \frac{m_0 \tau_0^2 c}{3!}\,.$$

Orbene, la (14) e la (13) rappresentano rispettivamente la lagrangiana e la equazione del moto per una particella polo-dipolare, del tipo di quelle considerate da BOPP, HÖNL, LUBANSKI, MATHISON e WEYSSENHOFF.

Un siffatto modello di elettrone, le cui equazioni del moto sono deducibili da una lagrangiana contenente le derivate prime della velocità, appare quindi rientrare come un caso particolare del modello di elettrone dato dalla nostra teoria.

Un altro metodo atto a trattare sistemi dinamici caratterizzati da lagrangiane dipendenti dalle derivate di ordine superiore delle coordinate è stato sviluppato, come abbiamo detto, da PAIS e UHLENBECK.

Ci proponiamo di applicarlo allo studio della equazione simmetrica non relativistica della nostra teoria.

Consideriamo il caso dell'elettrone libero dotato di un moto lungo una curva fissa, per cui avremo a che fare con un sistema a un sol grado di libertà. L'equazione del moto si scriverà:

$$F(D)q = 0\,, \tag{15}$$

dove $F(D)$ è una funzione pari (essendo il moto conservativo) dell'operatore $D = \mathrm{d}/\mathrm{d}t$. La lagrangiana sarà allora:

$$L = -\,qF(D)q\,. \tag{16}$$

Supposto $F(D)$ un polinomio pari di grado $2N$, potremo scrivere

$$F(D) = \prod_{\substack{1\\(j\neq i)}}^{N}{}_i\left(1 + \frac{D^2}{\omega_i^2}\right), \tag{17}$$

dove, senza essenzialmente limitare la generalità, supporremo le ω_i distinte.

Introdotte le nuove coordinate

$$Q_i = \prod_{\substack{1\\(j\neq i)}}^{N}{}_j\left(1 + \frac{D^2}{\omega_j^2}\right) q\,, \tag{18}$$

le (15) e (17) danno per le equazioni di moto:

$$(D^2 + \omega_i^2)Q_i = 0\,.$$

Il moto risulta cioè decomposto in N oscillazioni lineari di frequenza ω_i

La lagrangiana, nelle nuove variabili Q_j, può essere scritta sotto la forma.

$$\overline{L} = -\sum_{1}^{N}{}_j \eta_j Q_j(D^2 + \omega_j^2)Q_j\,, \tag{19}$$

dove le costanti η_j vanno determinate in modo che, a meno di una derivata rispetto al tempo di una funzione di q, $\dot{q}$ ... la quale dinamicamente non porta alcun contributo, $\bar{L}$ coincide con la L di (16). Usando la (16) e la (17) si ha:

$$L = -q\left[\sum_1^N{}_k \eta_k \left\{\prod_{\substack{1\\(j\neq k)}}^N{}_j\left(1+\frac{D^2}{\omega_j^2}\right)\right\}\right](D^2+\omega_k^2)\,q = -q\left[F^2(D)\sum_1^N{}_k \frac{\eta_k\omega_k^2}{1+D^2/\omega_k^2}\right]q$$

e confrontando con la (19) si trae:

$$\sum_1^N{}_k \frac{\eta_k\omega_k^2}{1+D^2/\omega_k^2} = \frac{1}{F(D)}\,, \tag{20}$$

dalla quale, scomponendo $1/F(D)$ in una somma di funzioni i cui numeratori sono indipendenti da D, si ha:

$$\eta_k = \frac{1}{\omega_k^4 F'(-\omega_k^2)}\,, \tag{21}$$

avendo indicato con $F'(-\omega_k^2)$ la costante $[\mathrm{d}F/\mathrm{d}(D^2)]_{D^2=-\omega_k^2}$

Con ciò le η_k risultano determinate.

Introdotte poi le variabili coniugate

$$P_i = \frac{\partial L}{\partial(D\dot{Q}_i)}\,,$$

dopo una trasformazione di contatto

$$P_i \to P_i(2|\eta_i|)^{\frac{1}{2}}, \qquad Q_i \to Q_i(2|\eta_i|)^{-\frac{1}{2}},$$

si ottiene l'hamiltoniana

$$H = \tfrac{1}{2}\sum_1^N{}_i (-1)^{i-1}(P_i^2+\omega_i^2Q_i^2)\,,$$

in cui il fattore $(-1)^{i-1}$ proviene dal fatto che le $F'(-\omega_k^2)$ nella (21), e quindi le η_k, sono di segno alternato.

Se, come nel nostro caso, $F(D)$ anzichè un polinomio di grado $2N$ è una funzione intera pari, per il teorema di Weierstrass sulla fattorizzazione potremo scrivere in generale

$$F(D) = \exp[f(D)]\prod_i\left(1+\frac{D^2}{\omega_i^2}\right), \tag{22}$$

con $\sum\omega_i^{-2}$ convergente e $f(D)$ funzione pari.

Si può dimostrare che se $f(D) = 0$ il procedimento precedente, valido per N finito, può essere esteso per $N \to \infty$.

Osservando che la nostra equazione simmetrica per il problema unidimensionale di una particella libera si può scrivere:

$$F(D)q = \frac{m_0}{\tau_0} D \sinh \tau_0 D \cdot q = 0 ,$$

la (22) diventa:

$$F(D) = m_0 D^2 \prod_1^\infty{}_i \left(1 + \frac{D^2}{\omega_i^2}\right),$$

con

$$\omega_k^2 = \frac{k^2\pi^2}{\tau_0^2} \qquad (k = 0, 1, 2, ...),$$

e la (20) si scrive:

$$\frac{\eta_0}{D} + \sum_1^\infty{}_k \frac{\eta_k \omega_k^2}{1 + D^2/\omega_k^2} = \frac{1}{F(D)} .$$

Tenendo presente lo sviluppo:

$$\frac{1}{(m_0/\tau_0) D \sinh \tau_0 D} =$$

$$= \frac{\tau_0^2}{m_0} \left\{ \frac{1}{\tau_0^2 D^2} - \frac{2}{\pi^2} \left[\frac{1}{1 + D^2/\omega_1^2} - \frac{1}{4} \frac{1}{1 + D^2/\omega_2^2} + \frac{1}{9} \frac{1}{1 + D^2/\omega_3^2} - ... + ... \right] \right\},$$

si ha:

$$(23) \qquad \begin{cases} \eta_0 = \dfrac{1}{m_0} , \\ \eta_k = (-1)^k \dfrac{2\tau_0^4}{m_0 \pi^4} \dfrac{1}{k^4} \end{cases} \qquad (k = 1, 2, ...).$$

Determinati tali valori si ha poi in forma esplicita la lagrangiana e l'hamiltoniana del nostro elettrone:

$$L = - \sum_0^\infty{}_k \eta_k Q_k (D^2 + \omega_k^2) Q_k ,$$

$$H = \tfrac{1}{2} P_0^2 + \tfrac{1}{2} \sum_1^\infty{}_j (-1)^j [P_j^2 + \omega_j^2 Q_j^2] .$$

L'applicazione del metodo esposto è resa assai difficile dal fatto che l'hamiltoniana ha una forma indeterminata; inoltre, per una particella sottoposta a un campo esterno, la separazione delle variabili è in generale impossibile. Tuttavia in qualche caso particolare è possibile ottenere dei risultati concreti.

Consideriamo, ad esempio, il caso in cui l'elettrone, descritto dall'equa-

zione simmetrica (8), sia soggetto all'azione di una forza elastica di richiamo $-kx$. In questo caso l'equazione del moto si scrive:

$$F(D)x = 0\,,$$

con

$$F(D) = \frac{m_0}{\tau_0} D \sinh(\tau_0 D) + k\,.$$

Posto $\tau_0 D = \xi$, $(\tau_0^2/m_0)k = a$, l'equazione che fornisce le frequenze fondamentali diviene:

$$F(D) = \xi \sinh \xi + a\,,$$

che, posto $\xi = iz$, dà

$$z \sin z = a\,.$$

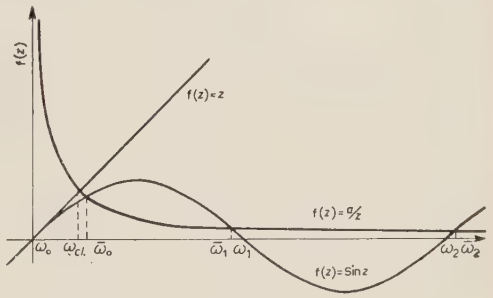

Fig. 1.

Come si può vedere, sia pure in modo qualitativo, dal grafico di fig. 1, rispetto all'equazione dell'elettrone libero $[(m_0/\tau_0)D \sinh \tau_0 D]x=0$, si ha uno spostamento pressochè irrilevante delle frequenze superiori (dato che a è piccolissimo) mentre è più sensibile lo spostamento della prima frequenza (nulla nell'equazione dell'elettrone libero); anzi si osserva un piccolo spostamento rispetto alla frequenza classica dell'oscillatore, come del resto era possibile affermare anche in base all'esame diretto dell'equazione del moto (cfr. § **5**).

7. – Relazione con un modello di elettrone esteso.

È interessante notare come la nostra equazione non relativistica dell'elettrone irraggiante sia la stessa che si ottiene considerando un opportuno modello di elettrone sferico a carica superficiale, quale è stato considerato da Bohm e Weinstein ([23]).

Detta infatti:

$$\varrho(\boldsymbol{x}, t) = ef(|\,\boldsymbol{x} - \boldsymbol{r}(t)\,|) \tag{25}$$

la densità di carica ($\boldsymbol{r}$ è il raggio vettore del centro di massa), con

$$\int f(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{x} = 1,$$

([23]) D. Bohm e M. Weinstein: *Phys. Rev.*, **74**, 1784 (1948).

avremo per la densità di corrente:

$$\boldsymbol{J}(\boldsymbol{x}, t) = \frac{e}{c}\,\dot{\boldsymbol{r}}(t) f(|\,\boldsymbol{x} - \boldsymbol{r}(t)\,|)\,.$$

Partendo da queste espressioni si può calcolare l'equazione del moto dell'elettrone che risulta:

$$m_m \ddot{\boldsymbol{r}} = \boldsymbol{F}_{(\text{reaz})} + \boldsymbol{F}_{(\text{est})}\,, \tag{26}$$

essendo $\boldsymbol{F}_{(\text{est})}$ la forza esterna, m_m la massa «meccanica» e $\boldsymbol{F}_{(\text{reaz})}$ la forza di reazione:

$$\boldsymbol{F}_{(\text{reaz})} = -4\pi e^2 \int_0^\infty \mathrm{d}\tau \int \frac{\mathrm{d}\boldsymbol{k}}{k}\,|f_k|^2 \exp\left[i\boldsymbol{k}\cdot s\right] \left\{ k\cos kc\tau\, \frac{\boldsymbol{k}\wedge(\dot{\boldsymbol{r}}(t-\tau))\wedge\boldsymbol{k}}{\boldsymbol{k}^2} - \right. \tag{27}$$

$$\left. - i \sin kc\tau\, \frac{\dot{\boldsymbol{r}}(t)}{\tau} \wedge \left(\boldsymbol{k}\wedge\dot{\boldsymbol{r}}(t-\tau)\right) \right\}$$

in cui $s = \boldsymbol{r}(t) - \boldsymbol{r}(t-\tau)$ e f_k è la k-esima componente dello sviluppo in integrale di Fourier della f:

$$f_k = \frac{1}{(2\pi)^{\frac{3}{2}}} \int f(|x|)\, \exp\,(-i\boldsymbol{k}\cdot\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\,,$$

Come hanno mostrato Bohm e Weinstein è notevole il fatto che la (26) non ha soluzioni autoaccelerate del tipo di quelle della equazione dell'elettrone di Dirac, ma invece soluzioni periodiche la cui frequenza dipende strettamente dalla distribuzione della carica sull'elettrone.

Specializzando poi la $f(x)$ assumendo

$$f(x) = \frac{\delta(x - R)}{4\pi R^2}\,,$$

Bohm e Weinstein riescono ad eseguire esplicitamente le integrazioni contenute in (27) arrivando al risultato

$$\boldsymbol{F}_{(\text{reaz})} = \frac{2}{3}\,\frac{e^2}{2R^2 c}\left[\boldsymbol{v}\left(t - \frac{2R}{c}\right) - \boldsymbol{v}(t)\right], \tag{28}$$

(nel calcolo si sono trascurati gli effetti magnetici).

Se si suppone pertanto che la massa sia tutta di origine elettromagnetica $m_0 = \frac{2}{3}\,(e^2/Rc^2)$ e si pone $\tau_0 = 2R/c$, la (28) si riduce esattamente all'equazione (3) non relativistica alle differenze finite della nostra teoria.

Notiamo ancora che alla stessa equazione era giunto anche Eliezer ([24])

([24]) C. J. Eliezer: *Proc. Camb. Phil. Soc.*, **46**, 198 (1950).

indipendentemente considerando un elettrone sferico con carica superficiale, per il quale già PAGE ([25]) aveva dato l'equazione del moto sotto la forma:

$$m_0 \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \frac{2}{3}\frac{e^2}{c^3}\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}t^2} + \frac{2}{3}\frac{e^2}{c^3 R}\sum_2^\infty{}^n (-1)^n \frac{(2R/c)^n}{(n+1)!}\frac{\mathrm{d}^n\dot{\boldsymbol{v}}}{\mathrm{d}t^n} = \boldsymbol{F}.$$

Infatti questa può scriversi formalmente (assumendo $m_0 = \frac{2}{3}(e^2/Rc^2)$):

$$\frac{m_0 c}{2R}\left\{1 - \exp\left(-\frac{2R}{c}\frac{\mathrm{d}}{\mathrm{d}t}\right)\right\}\boldsymbol{v}(t) = \boldsymbol{F},$$

che, per le note proprietà dell'operatore $\exp[-\lambda(\mathrm{d}/\mathrm{d}t)]$ e cioè

$$\exp\left[-\lambda\frac{\mathrm{d}}{\mathrm{d}t}\right] f(t) = f(t-\lambda),$$

risulta equivalente alla nostra equazione (3).

8. – Connessione con la teoria del campo elettromagnetico.

Le considerazioni del paragrafo precedente mostrano come, in approssimazione non relativistica, le equazioni del moto di un elettrone, quali risultano dalla nostra teoria, siano deducibili dalle equazioni maxwelliane del campo di cui l'elettrone può essere considerato come sorgente.

Che ciò sia possibile, data l'approssimazione non relativistica, è abbastanza pacifico ([26]). Come sappiamo dalle considerazioni di § **5** la cosa invece diventa alquanto difficile se si opera su equazioni del moto relativistiche. Era questa anzi una delle difficoltà maggiori della teoria dell'elettrone di Lorentz.

Ci proponiamo ora di mostrare come effettivamente la nostra equazione relativistica (4) dell'elettrone irraggiante possa essere dedotta dalle equazioni del campo elettromagnetico.

A tale scopo useremo un metodo dovuto a RZEWUSKI ([27]) che si basa sui seguenti principi:

([25]) L. PAGE: *Phys. Rev.*, **9**, 376 (1918).

([26]) Vogliamo notare che, seguendo un ordine di idee analogo a quello da noi esposto in questo paragrafo, G. HÖLER (*Ann. der Phys.*, **9**, 91 (1951)) ha mostrato che l'equazione non relativistica di MÖGLICH e ROMPE (cfr. § **4**) come pure quella non relativistica (3), già considerata da ELIEZER, possono essere dedotte dalla teoria del campo elettromagnetico secondo la formulazione di F. BOPP: *Ann. der Phys.*, **42**, 573 (1943); *Zeits. f. Naturf.*, **1**, 53 (1946).

([27]) J. RZEWUSKI: *Acta Phys. Polonica*, **9**, 203 (1953).

a) le equazioni del campo vengono dedotte, mediante principio variazionale, da una lagrangiana nelle derivate del 1° ordine delle componenti del campo;

b) allo scopo di evitare divergenze, le particelle sorgenti del campo vengono considerate come puntiformi ma interagenti col campo stesso attraverso un'azione a distanza (interazione di tipo non locale);

c) il passaggio dalle equazioni del campo a quelle del moto delle particelle si consegue esprimendo le soluzioni $A_\mu(x)$ come funzionali delle funzioni $\xi_\mu(\tau_n)$, $\dot{\xi}_\mu(\tau_n)$, ..., essendo $\xi_\mu(\tau_n)$ le equazioni parametriche della linea d'universo di una particella. Le equazioni del campo elettromagnetico possono, come è noto, ricavarsi dal principio variazionale

$$\delta \int L\left[x,\, A_\mu(x),\, \frac{\partial A_\mu(x)}{\partial x_\nu}\right] \mathrm{d}x = 0\,, \tag{29}$$

quando si prenda

$$L = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + A_\mu \tilde{s}_\mu\,,$$

essendo $\tilde{s}_\mu$ la sorgente del campo e

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\,.$$

Infatti dalla (29) si deduce

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = \tilde{s}_\nu\,,$$

che, con la condizione supplementare $\partial A_\mu/\partial x_\mu = 0$ sono equivalenti alla

$$\Box A_\nu = \tilde{s}_\nu\,. \tag{30}$$

La soluzione generale di questa equazione può scriversi:

$$A_\nu(x) = \int G(x - x')\tilde{s}_\nu(x')\mathrm{d}x'\,, \tag{31}$$

in cui $G(x - x')$ è una funzione di Green della (30) determinata parzialmente dalla condizione:

$$\Box G(x - x') = \delta(x - x')\,. \tag{32}$$

Il nostro problema consiste quindi nel trovare per $\tilde{s}_\nu$ una funzione atta a

descrivere correttamente le particelle, vale a dire tale che le equazioni del campo risultino equivalenti alle equazioni del moto delle particelle stesse.

Come è noto, per particelle puntiformi e a interazione locale, si suole porre:

$$s_\nu(x) = \sum_n e_n \int_{-\infty}^{\infty} \delta(x - \xi^n)\dot{\xi}^n_\nu \, \mathrm{d}\tau_n \, .$$

Noi, analogamente a quanto si fa nelle teorie ad interazione non locale, assumeremo:

$$\tilde{s}_\nu(x) = \sum_n e_n \int_{-\infty}^{\infty} F(x - \xi^n)\dot{\xi}^n_\nu \, \mathrm{d}\tau_n = \int_{-\infty}^{\infty} F(x - x')s_\nu(x') \, \mathrm{d}x', \tag{33}$$

ove la $F(x - x')$ è una funzione arbitraria dell'invariante $(x - x')^2$.

Si può allora facilmente constatare che il principio variazionale (29) può scriversi:

$$\delta \int L \, \mathrm{d}x = \delta \tfrac{1}{2} \sum_n \sum_m e_n e_m \int_{-\infty}^{\infty} \mathrm{d}\tau_n \int_{-\infty}^{\infty} \dot{\xi}^n_\mu \tilde{G}(\xi^n - \xi^m) \, \dot{\xi}^m_\mu \, \mathrm{d}\tau_m = 0 \, , \tag{34}$$

ove

$$\tilde{G}(x - x') = \iint F(x - x'') \, \mathrm{d}x'' G(x'' - x''') \mathrm{d}x''' \, F(x''' - x') \, . \tag{35}$$

È questo il principio di FOKKER ([28]) nella sua forma modificata datagli da FEYNMAN ([29]).

In esso manca il consueto termine in m_0 per il fatto che l'intera massa del l'elettrone viene considerata di origine elettromagnetica.

Indicando con $\tilde{A}_\nu$ il potenziale:

$$\tilde{A}_\nu(x) = \int F(x - x')A_\nu(x') \, \mathrm{d}x' = \sum_n e_n \int_{-\infty}^{\infty} \tilde{G}(x - \xi^n)\dot{\xi}^n_\nu \, \mathrm{d}\tau_n \, , \tag{36}$$

([28]) A. P. FOKKER: *Zeits. f. Phys.*, **58**, 886 (1929); *Physica*, **12**, 145 (1932).

([29]) R. P. FEYNMAN: *Phys. Rev.*, **74**, 939 (1948). La modifica apportata da FEYNMAN al principio variazionale di Fokker, consiste sostanzialmente nell'introduzione di un opportuno fattore $G(\xi^n - \xi^m)$ nell'integrale (34) al posto della funzione di Dirac $\delta(\xi^n - \xi^m)$ che compare nella formulazione originale del principio di Fokker per l'elettrone puntiforme. Per quel che riguarda le conseguenze fisiche di tale modifica rimandiamo ai lavori di FEYNMAN.

e ponendo

$$L_n = \sum_m e_m \int_{-\infty}^{\infty} \dot{\xi}_\mu^n \tilde{G}(\xi^n - \xi^u)\, \dot{\xi}_\mu^m \mathrm{d}\tau_m = \dot{\xi}_\mu^n \tilde{A}_\mu(\xi^n)\,, \tag{37}$$

il principio (34) si scrive:

$$\delta \tfrac{1}{2} \sum_n e_n \int_{-\infty}^{\infty} L_n \mathrm{d}\tau_n = 0\,, \tag{38}$$

che dà le equazioni di moto della n-esima particella:

$$e_n \sum_m \tilde{F}_{\mu\nu}^m(\xi^n)\dot{\xi}_\nu^n = 0\,, \tag{39}$$

con

$$\tilde{F}_{\mu\nu}^m(\xi) = \frac{\partial \tilde{A}_\nu^m(\xi)}{\partial \xi_\mu} - \frac{\partial \tilde{A}_\mu^m(\xi)}{\partial \xi_\nu}\,. \tag{40}$$

Denotando con

$$F_{\mu\nu}^{(\mathrm{est})}(\xi^n) = \sum_{m \neq n} \tilde{F}_{\mu\nu}^m(\xi^n)$$

il campo esterno, e con

$$F_{\mu\nu}^{(\mathrm{self})}(\xi^n) = \tilde{F}_{\mu\nu}^n(\xi^n)$$

quello proprio dell'elettrone, le (39) si scrivono anche:

$$e_n \left\{ F_{\mu\nu}^{(\mathrm{self})}(\xi^n) + F_{\mu\nu}^{(\mathrm{est})}(\xi^n) \right\} \dot{\xi}_\nu^n = 0\,.$$

Inoltre Rzewuski ha mostrato che, introdotto il momento

$$P_\nu = -\int_{-\infty}^{\sigma} \tilde{F}_{\mu\nu} s_\mu \mathrm{d}x$$

dell'insieme di particelle, questo è sempre nullo e risulta dato dalla somma:

$$P_\nu = P_\nu^{(\mathrm{self})} + P_\nu^{(\mathrm{int})}$$

del momento proprio complessivo delle particelle

$$P_\nu^{(\mathrm{self})} = \sum_n P_\nu^n = \sum_n e^n \int_{-\infty}^{\tau_n^\sigma} \mathrm{d}\tau_n \tilde{F}_{\mu\nu}^n(\xi^n)\dot{\xi}_\mu^n$$

(τ_n^σ è il valore di τ_n pel quale la linea d'universo della n-esima particella interseca la superficie σ) e del momento dovuto alle interazioni reciproche fra le varie particelle:

$$P_\nu^{(\mathrm{int})} = \sum_{n \neq m} \sum e_n \int_{-\infty}^{\tau_n^\sigma} \mathrm{d}\tau_n \tilde{F}_\mu^m(\xi^n)\dot{\xi}_\mu^n ;$$

inoltre vale la relazione:

$$\frac{\mathrm{d}P_\nu^n}{\mathrm{d}\tau_n^\sigma} = e_n F_{\mu\nu}^{(\mathrm{self})}(\xi^n)\dot{\xi}_\mu^n ,$$

che ci assicura che la massa definita dalle equazioni di moto è la stessa che compare nell'espressione del momento: con ciò viene ad essere superata una delle più gravi deficienze della teoria di Lorentz (cfr. § 2).

Vogliamo ora mostrare come sia possibile definire un fattore $\tilde{G}(\xi - \xi')$ tale che le equazioni del moto (39) che derivano dalle equazioni del campo elettromagnetico coincidano con quelle relativistiche della teoria da noi sviluppata per il moto dell'elettrone irradiante.

Tenendo infatti presente le (36), le (39) e le (40), basterà che per l'equivalenza suddetta sia verificata la condizione:

$$(41) \qquad e^2 \int_{-\infty}^{\infty} \mathrm{d}\tau' \left\{ \frac{\partial \tilde{G}(\xi - \xi')}{\partial \xi_\mu} u_\nu(\tau') - \frac{\partial \tilde{G}(\xi - \xi')}{\partial \xi_\nu} u_\mu(\tau') \right\} u_\nu(\tau) =$$

$$= \frac{m_0}{\tau_0} [u_\mu(\tau - \tau_0) + \frac{u_\mu(\tau) u_\nu(\tau)}{c^2} u_\nu(\tau - \tau_0)],$$

dove

$$\xi = \xi(\tau), \quad \xi' = \xi(\tau') \qquad \text{e} \qquad u_\mu(\tau) = \dot{\xi}_\mu(\tau), \quad u_\mu(\tau') = \dot{\xi}_\mu(\tau') .$$

Ammessa l'esistenza di più funzioni $\tilde{G}$ che soddisfano le equazioni precedenti, osserviamo che esse corrisponderanno a modelli di elettroni che si differenziano unicamente per i loro moti interni o, se vogliamo, per la loro struttura; evidentemente essi saranno dotati dello stesso moto macroscopico ([30]).

Vogliamo ora mostrare come si possa arrivare a una determinazione approssimata del fattore $\tilde{G}(\xi - \xi')$.

Dato che la funzione $\tilde{G}$ sarà praticamente diversa da zero solo per distanze $|\xi - \xi'| < \lambda$, essendo λ la lunghezza fondamentale, potremo sviluppare le varie funzioni che compaiono in (41) in serie di potenze del parametro

$$r = (\tau' - \tau)c .$$

([30]) Si osservi che l'imporre la condizione (41) equivale ad assumere che il campo proprio dell'elettrone sia dato dal tensore emisimmetrico

$$F_{\alpha\beta}^{(\mathrm{self})} = \frac{m_0}{e\tau_0 c^2} [u_\alpha(\tau) u_\beta(\tau - \tau_0) - u_\alpha(\tau - \tau_0) u_\beta(\tau)] .$$

Dopo aver eseguito un'integrazione per parti si ottiene:

$$\frac{e^2}{2c^2}\int \tilde{G}(-r^2)\mathrm{d}r\cdot\dot{u}_\alpha - \frac{2}{3}\frac{e^2}{c^3}\int \tilde{G}(-r^2)r\,\mathrm{d}r\cdot\left(\frac{u_\alpha u_\beta}{c^2}\ddot{u}_\beta + \ddot{u}_\alpha\right) + \ldots =$$

$$= m_0\dot{u}_\alpha - \frac{m_0\tau_0}{2}\left(\frac{u_\alpha u_\beta}{c^2}\ddot{u}_\beta + \ddot{u}_\alpha\right) + \ldots .$$

Volendo che questa sia soddisfatta nell'approssimazione in cui si considerano solo i termini scritti esplicitamente, si dovrà avere:

$$\frac{e^2}{2c^2}\int \tilde{G}(-r^2)\mathrm{d}r = m_0\,, \qquad \frac{2}{3}\frac{e^2}{c^3}\int \tilde{G}(-r^2)\,r\,\mathrm{d}r = \frac{m_0\tau_0}{2}\,,$$

e ricordando le espressioni di:

$$m_0 = \frac{2}{3}\frac{e^2}{Rc}\,, \qquad \tau_0 = \frac{2R}{c}\,,$$

si avrà:

$$\int \tilde{G}(-r^2)\,\mathrm{d}r = (4/3)\,R\,, \qquad \int \tilde{G}(-r^2)r\,\mathrm{d}r = 1\,.$$

Si verifica immediatamente che, nell'approssimazione in cui ci siamo posti, il fattore $\tilde{G}(-r^2)$ può essere preso eguale a:

$$\tilde{G}(-r^2) = \delta(-r^2+\lambda^2)$$

con

$$\lambda = \frac{3}{8}R\,.$$

Ciò corrisponde ad assumere [31]:

$$\tilde{G}(x-x') = \delta(-\sigma^2+\lambda^2) \tag{42}$$

con $\sigma^2 = |x-x'|^2$.

Osserviamo ancora che in seguito a tale determinazione del fattore di forma, il potenziale del campo nella nostra approssimazione sarà dato da:

$$\tilde{A}_\alpha(x) = e\int_{-\infty}^{\infty}\tilde{G}(x-x')\dot{x}'_\alpha\,\mathrm{d}\tau' = e\int_{-\infty}^{\infty}\delta(-\sigma^2+\lambda^2)u_\alpha(\tau')\,\mathrm{d}\tau', \tag{43}$$

che nel caso elettrostatico dà

$$\varphi(r) = \frac{e}{\sqrt{r^2+\lambda^2}} \tag{44}$$

che si mantiene finito per $r\to 0$.

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[31] Cfr. F. Duimio: *Nuovo Cimento*, **11**, 326 (1954). Ricordiamo come la funzione $\delta(-\sigma^2+\lambda^2)$ sia stata usata per la descrizione dell'elettrone da A. Landé: *Phys. Rev.*, **76**, 1176 (1949); **77**, 814 (1950); e da E. Groschwitz: *Zeits. f. Naturf.*, **7a**, 458 (1952).

9. – Soluzioni interne: il momento magnetico anomalo dell'elettrone.

Abbiamo già osservato che la nostra equazione alle differenze finite (4) ammette delle soluzioni periodiche arbitrarie che si aggiungono a quelle che descrivono il moto dell'elettrone ([32]).

Così nel caso di assenza di forze esterne agenti sull'elettrone, questo, come abbiamo visto, si muove di moto rettilineo uniforme, corrispondentemente alla soluzione $u_\alpha(\tau) = \text{cost.}$; l'equazione del moto ammette però, oltre a questa, qualsiasi soluzione (evidentemente periodica di periodo τ_0) dell'equazione:

$$u_\alpha(\tau) - u_\alpha(\tau - \tau_0) = 0 \,. \tag{45}$$

Ci proponiamo ora di indagare se non è possibile attribuire un significato anche a queste soluzioni periodiche che, come abbiamo già osservato appaiono legate a moti «interni» dell'elettrone vale a dire a problemi concernenti la struttura di questo.

Consideriamo allo scopo una particolare soluzione periodica della (45) e precisamente la seguente:

$$\begin{cases} \dot{x} = -\beta c \sin 2\pi \dfrac{\tau}{\tau_0} \,, \\ \dot{y} = \beta c \cos 2\pi \dfrac{\tau}{\tau_0} \,, \\ \dot{z} = 0 \,. \end{cases} \tag{46}$$

Integrando e ricordando che $\tau_0 = 2R/c$, si ha:

$$\begin{cases} x = \beta \dfrac{R}{\pi} \cos 2\pi \dfrac{c}{2R} \tau \,, \\ y = \beta \dfrac{R}{\pi} \sin 2\pi \dfrac{c}{2R} \tau \,, \\ z = 0 \,. \end{cases} \tag{47}$$

Tali soluzioni corrispondono a un moto di rotazione uniforme dell'elettrone lungo un cerchio di raggio $a = \beta R/\pi$ e con frequenza $\nu = c/2R$.

Come è noto tale moto dà origine a un momento magnetico diretto lungo l'asse z e dato da:

$$\mu = \frac{1}{c} \pi a^2 e \nu = \frac{1}{3\pi} \frac{e^3}{m_0 c^2} \beta^2 \,. \tag{48}$$

Ricordiamo ora che le esperienze recenti ([33]) hanno provato l'esistenza di

([32]) Cfr. P. Caldirola: *Nuovo Cimento*, **11**, 108 (1954).

([33]) J. E. Nafe e E. B. Nelson: *Phys. Rev.*, **73**, 718 (1948).

un momento magnetico «anomalo» dell'elettrone il cui valore numerico risulta eguale a $0{,}00114_5 \mu_B$.

Tale momento anomalo è stato previsto dall'elettrodinamica di SCHWINGER [34] che porta al seguente valore del rapporto giromagnetico g_a associato a μ_a:

$$g_a = -\frac{\mu_a}{s\mu_B} = 2\left(\frac{\alpha}{2\pi} - 2{,}973\frac{\alpha^2}{\pi^2}\right), \tag{49}$$

essendo $s = \frac{1}{2}$ lo spin (in unità $\hbar$) dell'elettrone e $\alpha = e^2/\hbar c$ la costante di struttura fina. Si noti che in prima approssimazione (cioè trascurando il secondo termine in α^2 nell'espressione di g_a) il valore del momento magnetico anomalo risulta:

$$\mu_a = \frac{1}{4\pi}\frac{e^3}{m_0 e^2}. \tag{50}$$

Confrontando le (47) con le (50) si vede che le due formule coincidono pur di prendere $\beta^2 = \frac{3}{4}$.

Si osservi che, in conseguenza di ciò, l'energia cinetica $E_{\text{cin}} = m_0 c^2 \cdot \left\{(1/\sqrt{1-\beta^2}) - 1\right\}$ legata al moto rotatorio interno (74) dell'elettrone risulta esattamente eguale a $m_0 c^2$, vale a dire alla energia intrinseca dell'elettrone; in altri termini l'energia intrinseca della particella quale appare nel suo moto macroscopico trae la sua origine dal moto interno. Questo fatto può essere ulteriormente compreso se si tiene presente che, dal punto di vista della teoria del campo, l'energia cinetica interna della particella può essere ricondotta all'energia del campo elettromagnetico proprio dell'elettrone.

Può sembrare a prima vista strano che la spiegazione dell'esistenza del momento magnetico anomalo dell'elettrone, la quale viene giustamente considerata come uno dei più brillanti successi della odierna teoria quantistica dei campi, possa anche essere conseguita, come qui si è mostrato, per mezzo di considerazioni classiche.

La ragione di questo fatto appare però chiara se si osserva che nell'espressione (50) del momento magnetico anomalo, che deriva direttamente dalla prima approssimazione della formula (49) di SCHWINGER, non è contenuta la costante di Planck: il contributo di prima approssimazione al momento magnetico anomalo dell'elettrone non è cioè un effetto essenzialmente quantistico. Tale è invece il contributo dovuto alla seconda approssimazione, per il quale sarebbe quindi illusorio tentare una spiegazione classica [35].

[34] J. SCHWINGER: *Phys. Rev.*, **73**, 416 (1948); R. KARPLUS e A. KLEIN: *Phys. Rev.*, **85**, 972 (1952).

[35] Queste osservazioni possono servire, a nostro parere, a chiarire una caratteristica difficoltà che si presenta nelle teorie dei campi ad interazione non locale. In

10. – Conclusioni.

La teoria dell'elettrone che abbiamo esposto si presenta interessante per il fatto che in essa vengono superate alcune tipiche difficoltà delle precedenti teorie di Lorentz e di Dirac. Notevole è il risultato che l'equazione relativistica del moto dell'elettrone irraggiante può essere ricavata dalle equazioni del campo elettromagnetico pur di considerare le sorgenti « estese » nel senso delle teorie ad interazione non locale; di particolare interesse è pure il risultato che essa riesce a dare una spiegazione semplice del momento magnetico anomalo dell'elettrone che viene pertanto presentato sotto una nuova luce.

Dal punto di vista fisico però ogni teoria classica si presenta come il presupposto per una corrispondente teoria quantistica che alla prima si appoggi nello spirito del principio di corrispondenza di Bohr. Si tratterebbe quindi di sottoporre la teoria sviluppata al processo di quantizzazione, programma questo piuttosto difficile, come si comprende facilmente tenendo presente le complicazioni che già si presentano nel tentativo di dare della nostra teoria una formulazione hamiltoniana.

Vogliamo solamente notare a questo proposito come sia da aspettarsi che l'equazione di Schrödinger per il nostro elettrone irraggiante ammetta delle soluzioni stazionarie corrispondenti ai moti classici « interni »: queste soluzioni dovrebbero corrispondere da un punto di vista quantistico ad uno spettro di autovalori dell'operatore hamiltoniano interpretabile come spettro di masse per una certa classe di particelle elementari.

Desideriamo ringraziare il dott. R. Cirelli per la sua apprezzata collaborazione nello studio di alcune delle questioni trattate in questo lavoro.

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queste teorie la carica è conservata sul contorno σ che racchiude un dominio Ω finito dello spazio-tempo, le cui dimensioni lineari sono dell'ordine di λ, mentre localmente non vale l'equazione di continuità. Ciò significa che entro Ω esistono delle correnti elettriche « interne »: a queste può attribuirsi l'esistenza di un momento magnetico dell'elettrone, il cui ordine di grandezza, in base a considerazioni puramente dimensionali, sarà dato da $\mu \simeq e\lambda = e^3/2m_0c^3$, vale a dire dell'ordine del momento magnetico anomalo.

SUMMARY

In this paper we examine the main difficulties connected with a classical theory of the electron interacting with the electromagnetic field, expecially and particularly those contained in the theories of Abraham-Lorentz and Dirac-Eliezer. Accepting the necessity of the introduction, still in the classical relativistic electrodynamics, of a

fundamental length, which has been proposed since long time by many people, we propose a finite-difference equation, relativistically invariant, which seems to eliminate the above said difficulties. Some applications of this equation to simple problems are discussed and a general lagrangian formulation of the theory is sketched, in which we use the methods of Ostrogradski and of Pais-Uhlenbeck. The enormous difficulties of this formulation, which should be the basis for a quantization, are emphasized. Then we state some connection between the present theory and others, namely with non-local field theories and with the « Pol-Dipol Teilchen » of HÖNL *et al.* The non relativistic approximation of our equation is shown to be equivalent with some models that have been proposed by various authors, and particularly with a model of spheric electron having the charge spread out on its surface. Finally we show how our theory could be inserted in the frame of a general theory of the electro-magnetic field interacting with its own sources, in which the mass of the electrons is completely of e.m. nature, without violating the principle of relativity. The last interesting consequence is the possibility of a classical interpretation of the anomalous magnetic moment of the electron, connected with the « microscopical solutions » describing the behaviour of the electron in small regions (of the order of the fundamental length) of space-time. This « internal structure » of the electron could play a very important role in a future quantized theory, where it could give perhaps rise to a mass spectrum of elementary particles.

Dosage de radioéléments par la distribution des intervalles entre désintégrations. Application au RdTh.

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(ricevuto il 17 Agosto 1954)

Résumé. — On compte les couples de désintégrations successives $\mathrm{Tn} \xrightarrow{\alpha} \mathrm{ThA} \xrightarrow{\alpha}$. Le détecteur utilisé est un compteur proportionnel. Un circuit spécial sélectionne les intervalles inférieurs à un temps donné. Le choix du temps optimum est discuté. La plus petite quantité de RdTh mesurable par la méthode décrite est celle en équilibre avec environ $3 \cdot 10^{-9}$ g de Thorium. Le rapport admissible entre les activités en α et en couples $\mathrm{Tn} \xrightarrow{\alpha} \mathrm{ThA} \xrightarrow{\alpha}$ est examiné.

Introduction.

Lorsqu'une source contient des radioéléments en filiation, les désintégrations observées ne sont pas statistiquement indépendantes. Si le noyau formé par une de ces désintégrations est lui-même instable, sa désintégration sera observée au cours de la même mesure, avec une probabilité d'autant plus grande que sa vie est courte par rapport au temps de la mesure. Dans l'émulsion photographique, cette corrélation se présente sous l'aspect bien connu d'étoiles dues aux désintégrations successives d'un même noyau. Dans le présent travail, nous décrivons une méthode de mesure des radioéléments naturels basée sur la corrélation dans le temps des désintégrations α.

Le problème particulier qui nous occupe est la mesure de très faibles activités α des isotopes du Th présents dans les sédiments océaniques. Ce problème a déjà été résolu par la méthode photographique [1,2]. Cette méthode présente des inconvénients d'ordre pratique quand il faut l'appliquer à de longues séries de mesures. L'observation microscopique est laborieuse et les solutions

(1) N. Isaac et E. Picciotto: *Nature*, **171**, 742 (1953).
(2) E. Picciotto et S. Wilgain: *Nature*, **173**, 632 (1954).

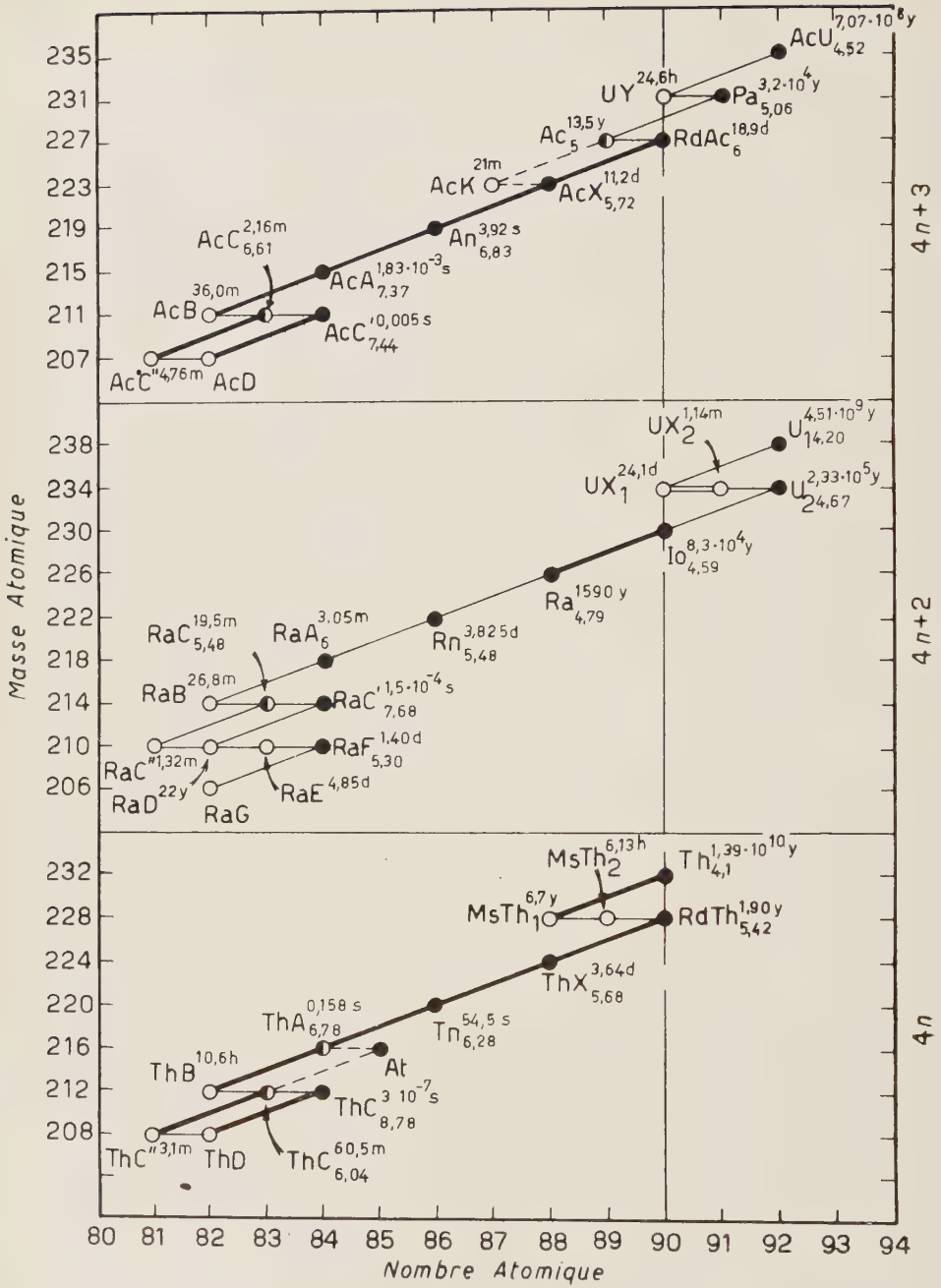

Fig. 1.

introduites dans l'émulsion doivent répondre à des conditions physico-chimiques assez strictes qui compliquent la partie chimique du travail. D'autre part, notre méthode permet de discriminer plus aisément le RdTh du RdAc.

Principe.

La fig. 1 représente les trois familles radioactives naturelles. Les isotopes du Th se trouvent sur la droite $Z = 90$. Le dosage direct porte sur le RdTh et le RdAc; la teneur en ^{232}Th peut être déduite de celle en RdTh s'il y a équilibre radioactif de ces deux isotopes; on trouve la teneur en Io par différence.

La mesure se fait un certain temps après la séparation chimique du Th, quand les descendants du RdTh et du RdAc ont recru en quantité suffisante; on peut éventuellement attendre l'équilibre, atteint au bout de quelques semaines. On observe les désintégrations renforcées sur la fig. 1.

La meilleure estimation des quantités de RdTh et RdAc est en principe celle qui résulte d'une analyse de la fonction d'auto-corrélation de la distribution dans le temps des α observés.

En pratique, on peut souvent se contenter d'une analyse plus simple. Les couples de désintégrations α successives, dont l'intervalle moyen est le plus court sont Tn $\xrightarrow{\alpha}$ ThA $\xrightarrow{\alpha}$ (0,23 s) pour les descendants du RdTh et An $\xrightarrow{\alpha}$ AcA $\xrightarrow{\alpha}$ ($2.6 \cdot 10^{-3}$ s) pour les descendants du RdAc. Connaissant le rendement du comptage et le temps écoulé depuis la séparation chimique des isotopes du Th, on peut calculer à partir du nombre de ces couples les teneurs en RdTh et RdAc.

Outre ces couples « réels », on observera un certain nombre de couples « fortuits », dus à des désintégrations rapprochées de deux noyaux distincts. La correction peut être estimée en analysant la distribution des intervalles. Mais quand la correction n'est pas excessive, on peut se contenter d'une analyse encore plus simple, faisant intervenir le nombre total d'α observés et le nombre d'intervalles inférieurs à un temps τ convenablement sélectionné. On choisit ce temps de manière à rendre minimum l'erreur relative sur le nombre de couples réels (voir Appendice).

Si l'activité totale de la source est telle que l'intervalle moyen entre deux désintégrations indépendantes est du même ordre de grandeur que l'intervalle moyen du couple réel, cette analyse devient inefficace.

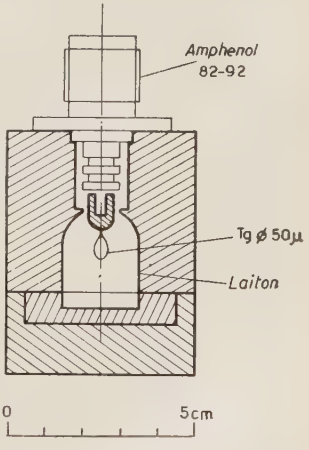

Fig. 2.

Compteur.

Pour discriminer entre particules α et β, nous avons utilisé comme détecteur, un compteur proportionnel sans fenêtre à circulation d'Argon (fig. 2) ([3]).

([3]) D. Taylor: *Nucleonics*, **11**, 3, 40 (1953).

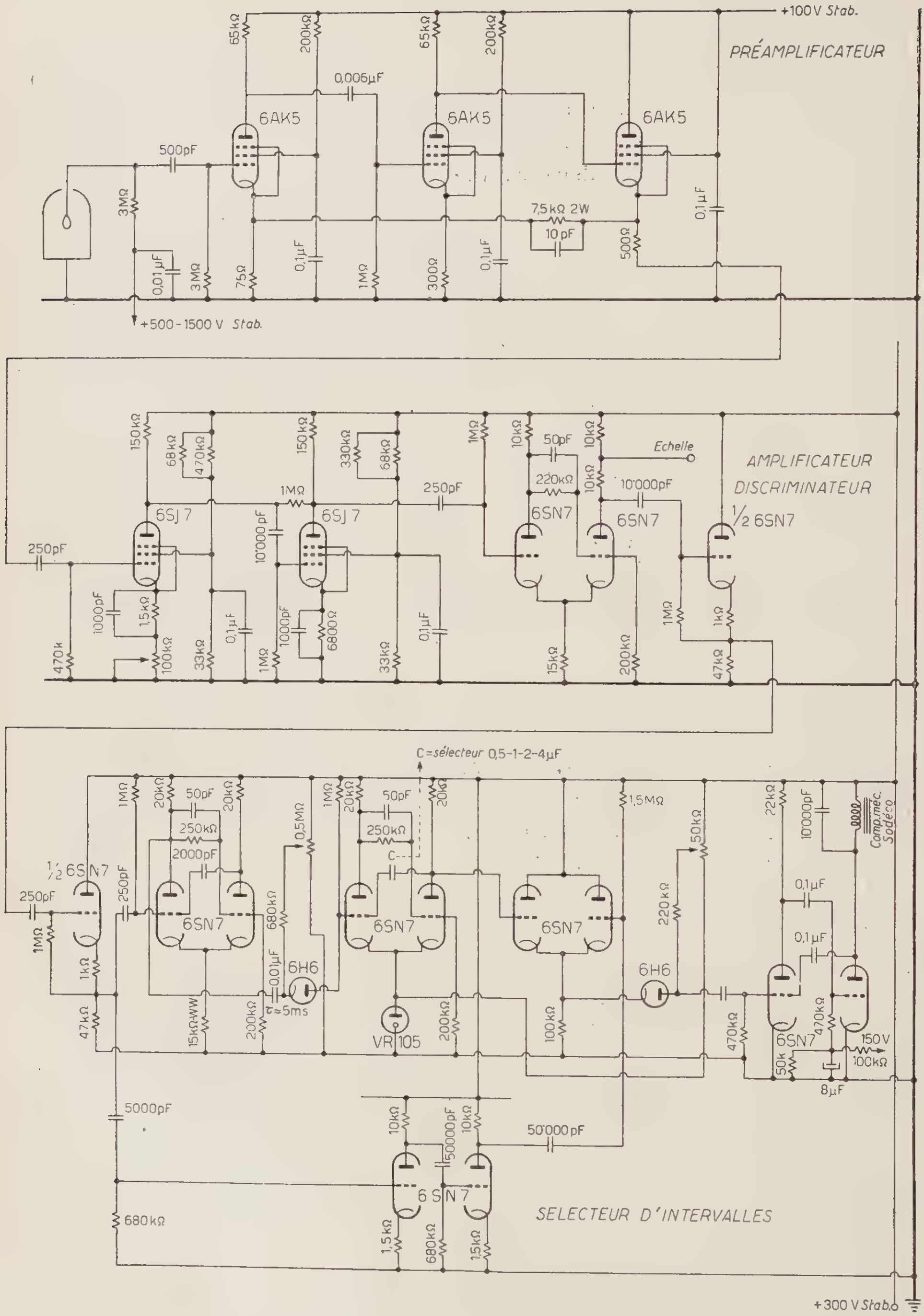
+100 V Stab.
PRÉAMPLIFICATEUR
65kΩ
200kΩ
0,006μF
6AK5
500pF
3MΩ
0,01μF
3MΩ
75Ω
0,1μF
1MΩ
300Ω
0,1μF
7,5kΩ 2W
10 pF
500Ω
0,1μF
+500-1500 V Stab.
150kΩ
68kΩ
470kΩ
6SJ7
1MΩ
10'000pF
330kΩ
250pF
1MΩ
10kΩ
50pF
220kΩ
Echelle
10'000pF
AMPLIFICATEUR
DISCRIMINATEUR
½ 6SN7
6SN7
1000pF
1,5kΩ
100kΩ
470k
33kΩ
6800Ω
15kΩ
200kΩ
1kΩ
47kΩ
C=sélecteur 0,5-1-2-4μF
20kΩ
250kΩ
2000pF
0,5MΩ
680kΩ
0,01μF
6H6
τ≈5ms
15kΩ-WW
VR 105
1,5MΩ
50kΩ
22kΩ
Comp. méc. Sodeco
0,1μF
150V
100kΩ
8μF
50k
5000pF
50'000 pF
50000pF
6 SN 7
SELECTEUR D'INTERVALLES
680kΩ
+300 V Stab.

Fig. 3.

Les dimensions résultent d'un compromis entre la nécessité d'un background faible (réduction de la surface) et d'une source suffisamment étendue. Il est important d'avoir un rendement élevé: la probabilité d'observer une désintégration α est égale au rendement, celle d'observer un couple est égale au carré du rendement. Pour faciliter la préparation des sources, nous avons travaillé en géométrie de 2π. Dans ces conditions on observe 50% des α et 25% des couples Tn-ThA émis par la source; dans le cas de Th en équilibre avec ses descendants, on observe 1 couple pour 12 α.

Circuits.

Nous décrivons uniquement les circuits pour la mesure du RdTh. La mesure du RdAc sera traitée dans un autre travail. La fig. 3 montre le schéma d'ensemble des circuits.

Les impulsions du compteur sont amplifiées par un préamplificateur du type 100 [4] et par un amplificateur-discriminateur classique. Les alimentations haute tension et basse tension des circuits sont stabilisées. La haute tension du compteur et le gain de l'amplificateur sont ajustés de façon à rendre l'appareil insensible aux β. Après amplification, les impulsions sont comptées par une échelle (activité totale); elles sont d'autre part injectées dans un ensemble à coïncidences retardées qui n'enregistre que les couples d'α séparés par un intervalle inférieur à un temps τ, choisi d'après les conditions expérimentales (voir Appendice). La fig. 4 représente le schéma de principe de ce circuit.

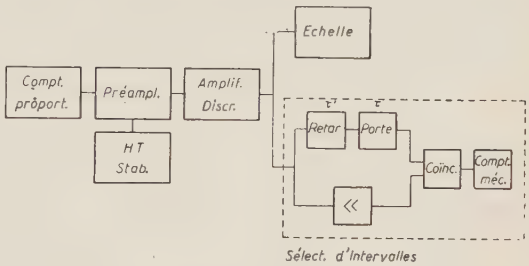

Fig. 4.

Chaque signal α, d'une part attaque après amplification l'une des entrées du circuit à coïncidences, d'autre part déclenche après un retard τ' une « porte » de durée τ appliquée à l'autre entrée du même circuit dont les coïncidences sont comptées. Le retard $\tau' \ll \tau$ permet d'éviter que toute impulsion α ne donne lieu à une coïncidence avec elle-même. Par un choix convenable de τ', on peut éliminer efficacement le couple An-AcA, tout en ne perdant qu'un faible pourcentage, connu, de RdTh. Comme il est facile de réétalonner τ et que $\tau' \ll \tau$, une très bonne stabilité de longue durée n'est pas obligatoire. Notre

[4] W. C. Elmore and M. Sands: *Electronics* (New York, 1949), p. 166.

circuit utilise deux « flip flop » en cascade. La mesure de τ se fait à l'aide d'un générateur d'impulsions dont la fréquence de répétition est variable. Quand l'intervalle entre les impulsions est inférieur à τ, aucun couple n'est compté; quand il devient égal à τ, la fréquence des couples devient brusquement égale à la moitié de la fréquence des impulsions.

Contrôles expérimentaux.

Les mesures ont été faites à 1500 V, le gain fixant le niveau de discrimination à 0,5 V. Le plateau du compteur, relevé avec une petite source de Po au centre du tiroir a, entre 900 et 1250 V, une pente inférieure à 2% aux 100 V. A 1150 V, le comptage diminue de moins de 3% quand on déplace la source dans un rectangle dont les côtes ont 0.5 cm parallèlement et 1,5 cm perpendiculairement au plan de la boucle. Le rendement mesuré à l'aide d'une source mince d'oxyde d'urane sur support de quartz est:

$$\eta = 0{,}505 \pm 0{,}023 .$$

Une source obtenue en évaporant sur un disque de quartz un volume de solution qui contient $1{,}55 \cdot 10^{-6}$ g de Th en équilibre avec ses descendants (*), donne les résultats suivants:

	Activité α totale par min.	Couples Tn-ThA par min.	Rapport
Mesuré	$1{,}17 \pm 0{,}03$	$0{,}094 \pm 0{,}012$	$12{,}5 \pm 1{,}5$
Calculé	1,14	0.095	12

Nous avons contrôlé la perte de Tn en activant le fil du compteur (— 600 V pendant 24 h) à l'aide d'une source de Tn en équilibre, donnant 5,5 α par minute. Le background avant l'activation était de $6{,}5 \pm 0{,}6$ α par heure et après, de $6{,}0 \pm 1{,}3$ α par heure; la perte de Tn est donc inférieure à 2% de la quantité formée (probabilité 99%). Les sources n'étaient pas recouvertes d'un film protecteur.

Sensibilité.

Le background en α du compteur utilisé est 6,5 α par heure. Pour une valeur de τ fixée à 0,467 s nous pouvons attendre 0,18 couples fortuits par

(*) Ce Thorium, vieux de plus de 50 ans, nous a été donné par le Dr. F. Koczy, de l'Institut Océanographique de Göteborg.

jour. En totalisant tous les essais de background nous arrivons à un couple observé en 3,5 jours.

Une source de $1{,}5 \cdot 10^{-7}$ g de Th en équilibre donne une activité α égale au background. Dans ces conditions, le temps nécessaire pour obtenir une

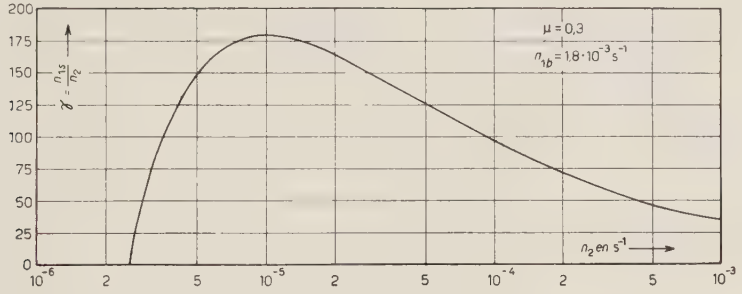

Fig. 5.

précision statistique donnée est le double du temps nécessaire en l'absence de background. Il en est de même pour le comptage de couples lorsque $f(\mu_0\,;\,(\tau/\vartheta)\ \text{opt}) = \sqrt{2}$, soit $\mu_0 = 0{,}3$ (voir Appendice et fig. 6). La plus petite activité en couples mesurable, définie par cette condition, est $n_{2\,\min} = 2{,}5 \cdot 10^{-6}$

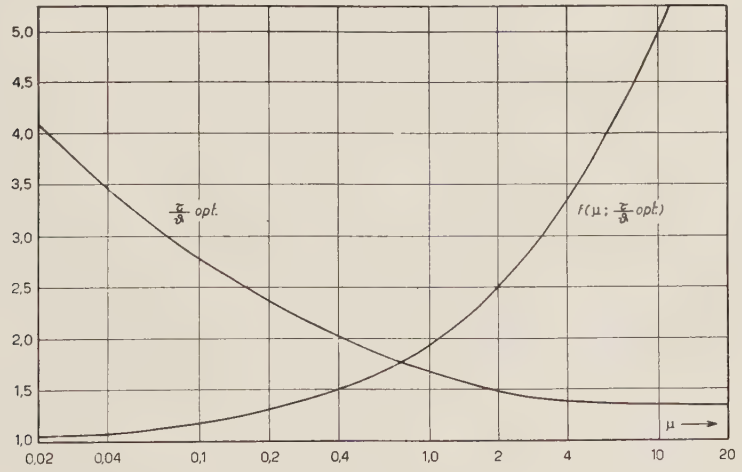

Fig. 6.

couples s^{-1} (= 0,2 par jour) qui correspond, au rendement de 50 %, à une quantité de RdTh en équilibre avec $3 \cdot 10^{-9}$ g de Th.

Abstraction faite de la difficulté d'immobiliser un compteur pendant des temps de l'ordre du mois, la sensibilité est comparable à celle de la méthode

photographique [2]; elle est largement supérieure à celle de la méthode classique par barbotage du Tn et comptage d'α qui permet de mesurer quelques microgrammes de Th. Une méthode plus sensible qui permet de détecter quelques dizièmes de microgramme est basée sur le comptage du Th(B+C) [5,6].

La courbe fig. 5 montre le rapport ν admissible, pour notre compteur, entre le nombre d'α de la source, n_{1s} et le nombre de couples n_2 émis par unité de temps dans le volume sensible du compteur.

En ce qui concerne ce rapport, notre méthode est inférieure à la méthode photographique.

Remerciements.

Nous remercions le Dr. E. Picciotto pour de précieux conseils et d'utiles discussions, les Professeurs P. Baudoux et G. P. S. Occhialini pour l'intérêt porté à ce travail, Madame S. Wilgain pour la préparation des sources et MM. A. De Waelheyns et A. Igiuni pour la construction des circuits.

Appendice

Soient t la durée de la mesure,

ϑ l'intervalle moyen des couples vrais (0,23 s pour Tn-ThA; $2{,}6 \cdot 10^{-3}$ s pour An-AcA),

n_1 le nombre d'α comptés par unité de temps,

n_2 le nombre de couples réels émis par unité de temps dans le volume sensible du compteur.

Les sélecteur de couples ne comptant que des intervalles de temps $\leqslant \tau$, le nombre moyen de couples réels comptés est

$$N' = n_2 t \, (1 - \exp [-\tau/\vartheta]) \, .$$

[5] F. G. Houtermans: *Sitzb. Heidelb. Akad. Wissen. Math.-Naturw.*, **2**, 123 (1951)
[6] J. C. Dalton, J. Golden, G. R. Martin, E. R. Mercer et S. J. Thomson: *Geochimica Acta*, **3**, 6, 272 (1953).

Comme il faut que l'activité α soit assez faible ($n_1\vartheta \ll 1$) pour permettre l'analyse des intervalles, le nombre de couples fortuits est

$$N'' = n_1 t(1 - \exp[-(n_1 - n_2)\tau]) \approx n_1(n_1 - n_2)\tau t\,,$$

τ étant choisi du même ordre que ϑ.

A partir du nombre total N de couples observés, on estime, si $n_2 \ll n_1$, le nombre de couples réels

$$N' = N - N''\,,$$

avec une erreur moyenne

$$\Delta N' = \sigma_N = \sqrt{\sigma_{N'}^2 + \sigma_{N''}^2} = \sqrt{N' + N''} \quad (\sigma = \text{écart quadratique moyen})$$

car N' et N'' sont indépendants; d'où en posant

$$\mu = \frac{n_1(n_1 - n_2)\vartheta}{n_2}\,,$$

l'erreur relative

$$\frac{\Delta N'}{N'} = \frac{1}{\sqrt{n_2 t}} \frac{\sqrt{1 - \exp[-\tau/\vartheta] + \mu}}{1 - \exp[-\tau/\vartheta]} = \frac{1}{\sqrt{n_2 t}} f\left(\mu;\frac{\tau}{\vartheta}\right),$$

la présence de couples fortuits obligeant de compter pendant un temps $f^2(\mu;\tau/\vartheta)$ fois plus long pour obtenir une même précision statistique.

La valeur de τ/ϑ qui rend $\Delta N'/N'$ minimum est solution de

$$\frac{\partial f}{\partial(\tau/\vartheta)} = 0 \quad \text{ou} \quad \mu = \frac{\exp[-\tau/\vartheta]\cdot(1 - \exp[-\tau/\vartheta])}{1 - \exp[-\tau/\vartheta] - 2(\tau/\vartheta)\exp[-\tau/\vartheta]}\,.$$

Les courbes de la fig. 6 donnent en fonction de μ la valeur optima de τ/ϑ ainsi que la valeur de $f(\mu;\tau/\vartheta)$ lorsqu'on y introduit τ/ϑ optimum.

L'activité n_1 comprend le background du compteur et l'activité de la source

$$n_1 = n_{1b} + n_{1s}\,.$$

L'erreur relative augmente rapidement quand μ croît, soit μ_0 la plus grande valeur acceptable. Le plus petit n_2 mesurable est ($n_1 \gg n_2$)

$$n_{2\,\min} = \frac{n_{1b}^2\vartheta}{\mu_0}$$

et le plus grand rapport ν admissible entre les activités en α et en couples de

la source

$$\nu = \frac{n_{1s}}{n_2} = \sqrt{\frac{\mu_0}{\vartheta n_2}} - \frac{n_{1b}}{n_2} \qquad (n_2 \geqslant n_{2\,\min}) .$$

Ce rapport, nul quand $n_2 = n_{2\,\min}$, passe par le maximum

$$\nu_{\max} = \frac{\mu_0}{4 n_{1b} \vartheta}$$

lorsque $n_2 = 4 n_{2\,\min}$.

RIASSUNTO (*)

Per mezzo di un contatore proporzionale si contano le coppie di disintegrazioni successive $\mathrm{Tn} \xrightarrow{\alpha} \mathrm{ThA} \xrightarrow{\alpha}$. Un circuito speciale seleziona gli intervalli inferiori a un tempo dato. Si discute la scelta del tempo ottimo. La minima quantità di RdTh misurabile col metodo descritto è quella in equilibrio con $\sim 3 \cdot 10^{-9}$ g di torio. Si esamina il rapporto ammissibile tra le attività in α e in coppie $\mathrm{Tn} \xrightarrow{\alpha} \mathrm{ThA} \xrightarrow{\alpha}$.

(*) *Traduzione a cura della Redazione.*

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Rôle of Exchange Forces in the Problem of Helium II.

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Summary. — Appropriate one-particle eigenfunctions are built up for excitons in liquid He which are quadratic in momentum and the exchange and « Coulomb » integrals for such eigenfunctions are studied. Independently, an analysis of experimental data concerning the energy absorbed per exciton in the interval from 0 °K to the λ-point shows that the exchange energy must on the average be small. It is then shown that the exchange integral (in contrast with the « Coulomb » integral) cannot have a constant value but must increase with temperature, unless some appropriate mechanism prevents it from doing so. Though not analyzing in detail such mechanism, a tentative is made to show that it must be the same which is responsible for the negative expansion between 1 °K and the λ-point. Other topics touched upon are effective masses, excitation energies, and the behaviour of a saturated phonon gas.

1. – Introduction.

In a very interesting paper of 1941 Bijl, De Boer and Michels [1] for the first time (at least as far as the author knows) introduced exchange forces in the study of He II.

They did so in an attempt to account for the form of the specific heat curve through the introduction of an excitation energy, possibly a function of the « superfluid » content of the liquid. Indeed an excitation energy seems to separate from the ground state those excitations which are quadratic in momentum and are prevalent above, say, 1 °K. To account for this energy gap is one of the tasks that may be attributed to exchange forces. However,

(1) A. Bijl, J. De Boer and A. Michels: *Physica*, **8**, 655 (1941).

examining the problem over again in greater detail as here will be done, we shall see that a deeper insight on the whole question can be gained by taking into consideration another effect that may be traced to exchange, i.e. the *negative expansion* which takes place between, say, 1 °K and the λ-point.

The statement that negative expansion can have anything to do with exchange forces requires some justification.

On very general lines (virial theorem) one can say that the density of a condensed body is the resultant of an equilibrium between cohesion forces and the kinetic energy of the constituent particles, so that any anomaly in the density should be attributed to one of these factors ([2]). In the case of He II the kinetic energy could hardly be thought of behaving anomalously: in fact, because of the large value it has already at 0 °K, the kinetic energy can change but very little (and presumably only increase) in going from 0 °K to the λ-point. Turning then to cohesion forces, we shall note that Einstein condensation requires the presence of a number of particles of the order of Avogadro's number in *one* state: a state of affairs that disappears abruptly at the λ-point (just as the anomaly in the expansion does). We are thus led to look for forces depending critically upon quantum states and in particular being sensitive to the existence of very large occupation numbers. This suggests immediately exchange forces.

Wit a *constant* value I for the exchange integral, that is a value independent of the states of the particles, the exchange energy would be ([3]) $\frac{1}{2}(N^2 - N_0^2)I$ where N is the total number of particles and N_0 that of « condensed » ones. We cannot expect quantitative agreement from any explanation of the density anomaly based on an expression like this one: to be sure, it increases with T, as the density does, but is *stationary* at the λ-point, where $\partial\varrho/\partial T$ reaches its maximum. From this simple fact we see that it is essential to get a clearer idea of the exchange integral and its dependence on the various parameters that may enter it.

2. – One-particle Eigenfunctions.

The first step is to set up appropriate eigenfunctions. This problem has been dealt with recently in particular by FEYNMAN ([4]) who, however has considered many-particle eigenfunctions. On broad lines it seems that Feynman's

([2]) Although this might be an oversimplified argument for intricate structures, as for instance water, one feels that it should work for such a (comparatively) simple, and indeed so « quantum-mechanical », liquid as liquid helium is.

([3]) At least in the ordinary approximation consisting in averaging over perturbed levels.

([4]) R. P. FEYNMAN: *Phys. Rev.*, **94**, 262 (1954).

eigenfunctions are the best ones for dealing with those excitations that are *linear* in momentum (phonons) whereas better results seem to be obtained for *quadratic* excitations with the simpler one-particle eigenfunctions studied in this paper (which, in turn, are not adequate for describing phonons). Since the anomalous expansion belongs to the region of temperature where quadratic excitations are dominant, we shall deal here with the latter ones only.

These eigenfunctions have been derived starting from the well known Einstein's idea that the motion of a molecule must be determined to a first approximation by the average field due to its neighbours. One must however carefully consider the fact that this average field is strongly dependent on correlations between the movements of the molecules. Overlooking this point, the author first considered the motion of an He atom in the field of its neighbours assumed fixed at their equilibrium positions. Main features of the calculation were: *a*) use of SLATER and KIRKWOOD's [5] formula for the potential energy W of two atoms at a distance r

$$W = 7.7 \cdot 10^6 \exp[-4.59r] - 1.49 \cdot 10^4 r^{-6}, \tag{1}$$

(Ångstrom units for lengths and 10^{-16} erg for energy are used); *b*) assumption that atoms in the liquid form a diamond lattice [6]; *c*) averaging over various directions to get a spherical potential well; *d*) use of KIMBALL and SHORTLEY's [7] numerical method to solve Schrödinger's equation.

The levels obtained by this method have *positive* energies [8]. The ground level ($n=0$, $l=0$) lies at $30.5 \cdot 10^{-16}$ erg, the next one ($n=0$, $l=1$) at 98.4 (both with about $50 \cdot 10^{-16}$ kinetic energy).

These impossible values mean simply that correlations strongly lower the potential energy of the average field in which every atom is moving, rendering difficult to predict what the form of the potential well should be. This would constitute a problem in itself — very likely worth con-

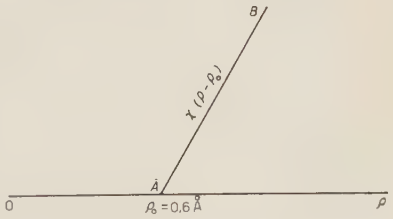

Fig. 1. - The simplified potential hole adopted for the motion of an atom in the average field of its neighbours. It is supposed to take account (with an appropriate value for χ) of the effect of correlation of movements. (ϱ distance from « equilibrium » position).

[5] J. C. SLATER and J. G. KIRKWOOD: *Phys. Rev.*, **37**, 682 (1931).

[6] This is only partly confirmed by recent work. See J. REEKIE and T. S. HUTCHINSON, *Phys. Rev.*, **92**, 827 (1953).

[7] G. E. KIMBALL and G. H. SHORTLEY: *Phys. Rev.*, **45**, 815 (1934).

[8] This fact is probably related to the inhability for He to form a solid (for which the assumption of fixed neighbours is less unsatisfactory) without the help of an external pressure.

sideration — but leaving it aside for the moment, the author choose to assume for the spherically symmetrical potential well the simplest possible form shown in Fig. 1: a plateau followed by a linear increase $\chi(\varrho - \varrho_0)$. The origin is of course the «equilibrium position» of the atom and the potential energy is arbitrarily taken zero at this point. The value of $\varrho_0 = OA = 0.6$ Å was suggested by the previously mentioned potential well, which shows a bend at about this distance. Thus one parameter only remains to be fixed, namely the slope χ of the rising portion. A value of $\chi = 40 \cdot 10^{-16}$ erg/Å has been chosen for reasons that will appear later.

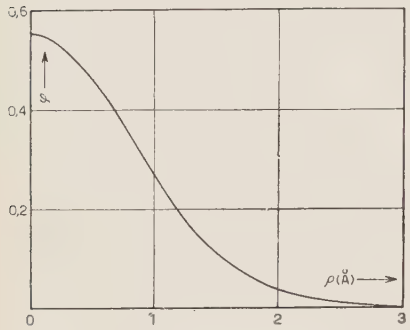

Fig. 2. – Ground state eigenfunction. (Potential hole Fig. 1).

Approximating the broken line OAB in the significant portion (about $0 < \varrho < 2$ Å) by a parabola and then applying 1st order perturbation theory, approximate eigenfunctions and eigenvalues are obtained which have been perfected by Kimball and Shortley's method.

Only the lowest state is of interest here, whose eigenfunction $\varphi(\varrho)$ is reported in Fig. 2 and in Table I. It is normalized so that

$$4\pi \int_0^\infty \varphi^2(\varrho)\varrho^2 \, d\varrho = 1 \,. \tag{2}$$

Table I.

ϱ	$\varphi(\varrho)$	ϱ	$\varphi(\varrho)$	ϱ	$\varphi(\varrho)$
0.0	0.553	1.4	0.135	2.8	0.00403
0.2	0.536	1.6	0.0908	3.0	0.00217
0.4	0.494	1.8	0.0587	3.2	0.00114
0.6	0.431	2.0	0.0367	3.4	0.000589
0.8	0.349	2.2	0.0220	3.6	0.000304
1.0	0.266	2.4	0.0130	3.8	0.00018
1.2	0.194	2.6	0.00733	4.0	0.00013

The corresponding energy is [9]

$$E_c = 32.6 \cdot 10^{-16} \text{ erg} \tag{3}$$

[9] Of course, eigenvalues are positive, but this has no particular significance in the present case since here we have arbitrarily shifted the zero of the energy scale.

with a kinetic energy $\mathcal{T}_c$

$$\mathcal{T}_c = 17.1 \cdot 10^{-16} \text{ erg} . \tag{3'}$$

Next level ($n = 0$, $l = 1$) lies some 22 units higher (corresponding to about the same number of degrees of temperature for a saturated Bose-Einstein perfect gas).

The « local » eigenfunction $\varphi(\varrho)$ itself is however not yet the appropriate one for describing the peculiar properties of liquid helium II. In fact some of these properties have lead to think of the eigenfunctions as spread over macroscopic spaces (or at least spaces of much more than atomic size). Thus the idea suggests itself of combining a number of the above eigenfunctions so as to fill up the space in which the atom can move, and the quasi-periodic structure of the liquid suggests putting

$$\psi(\boldsymbol{\rho}) = \sum_n c_n \varphi(|\boldsymbol{\rho} - \boldsymbol{r}_n|) \tag{4}$$

where $\boldsymbol{r}_n$ are constant vectors ending at the nodal points of an appropriate lattice.

Clearly, one should expect the fundamental state to correspond to c_n=const. To get excited states we have only to let the c_n vary, i.e. to be dependent on n. Choosing

$$c_n = C_n \exp[2\pi i \boldsymbol{K} \cdot \boldsymbol{r}_n] \tag{5}$$

it is easy to show that the functions

$$\psi_K(\boldsymbol{\rho}) = \sum_n C_K \exp[2\pi i \boldsymbol{K} \cdot \boldsymbol{r}_n] \varphi(|\boldsymbol{\rho} - \boldsymbol{r}_n|) \tag{6}$$

form an orthogonal set, provided only that the $\boldsymbol{K}$'s are subjected to the familiar conditions characteristic of waves in periodic structures ([10]), which practically amount to say that they form a continuum, with an upper limit (scarcely important in the present case, for temperature reasons).

With $K = 0$ we come back to the fundamental state. The spreading out of the eigenfunctions in this case must be attributed essentially to the overlapping of the eigenfunctions φ's themselves. Indeed it must be borne in mind that whereas in ordinary liquids, the quantum indeterminacy of position is very small in comparison of intermolecular distances, here the two are of the same order of magnitude.

([10]) In reaching this conclusion, only the overlapping of eigenfunctions belonging to nodes of the lattice which are *not* next neighbours has been neglected.

Normalization can be satisfied by putting.

$$|C_{\boldsymbol{K}}|^2 = N^{-1}(1 + 4\sigma \cos \pi K a)^{-1} . \tag{7}$$

Here N is the total number of nodes in the lattice (of atoms in the fluid), a is the distance between next neighbours (round 3.2 Å) and σ is the integral

$$\sigma = \int \varphi(|\boldsymbol{\rho} - O_1|)\, \varphi(|\boldsymbol{\rho} - O_2|)\, \mathrm{d}\boldsymbol{\rho} , \tag{8}$$

where O_1 and O_2 are two neighbouring nodes of the lattice.

The function $\cos \pi K a$ enters eq. (7) because one has to evaluate expressions like

$$S = \sum_\nu \exp [2\pi i \boldsymbol{K} \cdot (\boldsymbol{r}_\nu - \boldsymbol{r}_m)],$$

where ν enumerates the next-neighbouring nodes N_ν around a central one (M, at $\boldsymbol{r}_m$). Assuming these are four and are located at the vertexes of a regular tetrahedron centered at M, and calling $\boldsymbol{\tau}_\nu$ the unit vectors lying in the directions MN_ν, the above expression becomes

$$S = \sum_\nu \exp [2\pi i \boldsymbol{K} \cdot \boldsymbol{\tau}_\nu a] .$$

Now, on the average, among the projections of the four $\boldsymbol{\tau}_\nu$ on $\boldsymbol{K}$, two will be negative and two positive, with an average modulus very near $\frac{1}{2}a$. Thus we get

$$S = 2(\exp [\pi i K a] + \exp [-\pi i K a]) = 4 \cos \pi K a .$$

The physical meaning of $\psi_{\boldsymbol{K}}(\boldsymbol{r})$ is apparent from its structure: it represents a particle travelling with a constant speed along a rectilinear path, which however is, so to speak, « full of holes », regularly spaced. Thus the existence of the other particles is at least partly taken into account while writing one-particle eigenfunctions. This procedure appeared to the author to constitute the simplest step on the way that starts with free particles to end to the real state of affairs for particles in a liquid.

3. – Kinetic Energy and Effective Mass. Zero Point Energy.

We next ask for the kinetic energy associated with eigenfunctions (6). By use of the standard formula

$$\langle E_{\mathrm{Kin.}} \rangle_{\mathrm{Av.}} = -\int \psi^*_{\boldsymbol{K}} \frac{h^2}{8\pi^2 m} \nabla^2 \psi_{\boldsymbol{K}}\, \mathrm{d}\boldsymbol{r}$$

and considering that the φ's satisfy Schrödinger's equation with the potential already described, one finds without difficulties, for the excess kinetic energy above that of the fundamental state ($K=0$)

$$\Delta\mathcal{C} = -4(\mathcal{J} - \sigma\mathcal{C}_c)(1 - \cos \pi K a) . \tag{9}$$

Here $\mathcal{C}_c$ has the value (3'), σ is given by (8) and $\mathcal{J}$ (a negative quantity) is the integral

$$\mathcal{J} = \int \varphi(|\boldsymbol{\rho} - O_1|)[E_c - W(\varrho)]\,\varphi(|\boldsymbol{\rho} - O_2|)\,\mathrm{d}\boldsymbol{\rho} , \tag{10}$$

with E_c given by (3) and O_1, O_2 two neighbouring nodes of the lattice. For $\mathcal{C}_0$ itself, that is the kinetic energy for $K=0$, an expression

$$\mathcal{C}_0 = \frac{\mathcal{C}_c + 4\mathcal{J}}{1 + 4\sigma} \tag{11}$$

is found.

Numerically computed values (with $\chi = 40 \cdot 10^{-16}$ erg Å^{-1}) are as follows

$$\sigma = 0.0193 , \qquad \mathcal{J} = -0.442 \cdot 10^{-16} \text{ erg} , \qquad \mathcal{C}_0 = 14.2 \cdot 10^{-16} \text{ erg} . \tag{12}$$

The cosine appears in eq. (9) for a reason exactly similar to that explained in the previous Section. Replacing it by $1 - \frac{1}{2}\pi^2 K^2 a^2$ should make only a few per cent error. We have therefore

$$\Delta\mathcal{C} = -2\pi^2(\mathcal{J} - \sigma\mathcal{C}_c)a^2K^2 . \tag{13}$$

This formula is interesting in two respects. It shows a *quadratic* dependence on the wave number, that is on momentum, and enables one to write an expression for the *effective mass* μ. Indeed by simply equating $\Delta\mathcal{C}$ to $(1/2\mu)\hbar^2K^2$, one has

$$\mu = -\hbar^2/4\pi^2a^2(\mathcal{J} - \sigma\mathcal{C}_c) . \tag{14}$$

The quadratic dependence shows that our excitons have nothing to do with phonons, a feature that was already apparent from the structure of the eigenfunctions (6) themselves. They might prove instead an useful alternative for rotons. Indeed, whereas it is admissible to think of phonons, that is waves in a continuum, as the very lowest excitations of a liquid, it is much less satisfactory to assume that in such a gas-like liquid as He is, vortex

motions are setting in by increasing temperature *before* any translational movement has done so.

As regards the effective mass, we find for ^{4}He, introducing values (12) into (14)

$$\mu = 14 \cdot 10^{-24} \qquad (^4\text{He}) \tag{15}$$

(Mass of ^{4}He: $m_{\text{He}} = 6.64 \cdot 10^{-24}$). However, the *one*-particle eigenfunctions (6) on which the calculation of μ is based, should be valid also — in particular — for ^{3}He present as an impurity. Were moreover the walls of the potential hole to which φ refers infinitely steep, the eigenfunctions would not depend on m, and therefore μ (from eq. (14)) would turn out to be proportional to m. Assuming this is approximatively valid here, we should expect for ^{3}He an effective mass

$$\mu' = 10.5 \cdot 10^{-24} \qquad (^3\text{He}) \tag{15'}$$

which compares not too unfavourably with the values of the order of three times the atomic mass found experimentally ([11]) (mass of ^{3}He: $4.98 \cdot 10^{-24}$).

We shall see however (Sect. **6**) that the consideration of *exchange* (which of course does not apply to ^{3}He at *small concentrations*) is also effective in determining the apparent mass.

Concerning the zero-point energy, we should start by noting that this is not a too well defined quantity, as it contains a potential part whose value always depends on some conventions. To get a clear idea of it, let us write a balance in which it takes part. This can be achieved in various ways. We can start for instance by considering the « lattice energy » of the liquid, that is a fictitious potential energy W_0 (per atom) which would correspond to having the atoms located exactly at the nodes of the lattice. W_0 can be calculated from the interaction law (1), knowing numbers of neighbours and their distances around any atom. Utilizing the data of Reekie and Hutchinson (ref ([6])) a value of $-35 \cdot 10^{-16}$ erg/at. has resulted, practically coincident with that given by the London formulae ([12]) (diamond-type lattice). Clearly, between W_0 and the zero-point energy U_0 we should expect following relation to hold at any temperature

$$U_0 + \frac{1}{N}\int_0^T C \, \mathrm{d}T + \frac{1}{N} H_T = W_0 + \frac{3}{2} kT \, . \tag{16}$$

([11]) See J. G. Daunt: *Phil. Mag. Suppl.*, **1**, 258 (1952).
([12]) F. London: *Proc. Roy. Soc. Lond.*, A **153**, 581 (1936).

Here C is the (molal) specific heat, H_T the (molal) latent heat of vaporization, while the vapour has been attributed the energy of an ordinary perfect monoatomic gas.

Introducing known data ([13]), a value of about

$$U_0 = 22.5 \cdot 10^{-16} \text{ erg/at.} \tag{17}$$

is obtained, which is somewhat lower than usually stated, but agrees well with London's calculations (ref. ([12])).

To know what this result should be compared to, we note that since W_0 is the increase in potential energy per atom in going from a perfectly ordered lattice to a state of infinite separation of the atoms, we have to include in U_0 any energy contributing to this aim. This means that if the atoms could be described as belonging each to its own cell, half the (positive) potential energy of the motion within the cell should be added to the kinetic energy to make U_0. This would give, according to (3) and (3′)

$$U_0 = \tfrac{1}{2}(E_c + \mathcal{C}_c) = 24.8 \cdot 10^{-16} \text{ erg/at.}\,.$$

However, we have to combine the φ's, going over to eigenfunctions (6) and by so doing the kinetic energy of the fundamental state is lowered somewhat. As regards the potential energy, the author was unable to find a satisfactory way to calculate its change. This may prove perhaps a drawback of the present method, which might be characterised as a cellular method with unbounded cells. Assuming the potential energy not to be seriously affected, we should simply have an U_0 value lower than the above one by an amount $\mathcal{C}_c - \mathcal{C}_0 = 2.9$, making

$$U_{0\,\text{calc.}} = 21.9 \cdot 10^{-16} \text{ erg/at.}\,. \tag{17′}$$

Of course, the unknown constant χ — on which the numerical data (12) depend — was chosen just to achieve agreement between (17) and (17′), as it was thought that the data on U_0 were more reliable than those on effective mass.

Perhaps as much can be said that it is an encouraging feature of the present model that once χ has been adjusted to give the correct amount of zero-point energy it also gives an effective mass of the right order of magnitude. Indeed we should hardly expect anything more from so schematic an assumption as that embodied in Fig. 1 for the potential well.

([13]) Data for specific heat: H. C. KRAMERS, J. WASSCHER and C. J. GORTER: *Physica*, **18**, 329 (1952). Data for vaporization heat: J. KISTEMAKER: *Physica*, **12**, 281 (1946).

An alternative method of calculating the zero-point energy, which avoids the above difficulty concerning the potential energy, could be used if data on the distribution function of the atoms around a generical one at *absolute zero* were available. This would consist in computing the *effective* potential energy, starting from such a distribution and the interaction law (1). The same quantity with opposite sign would give the increase in potential energy when going from the liquid at 0 °K to the (rarefied) vapour. Putting this datum instead of the lattice energy in eq. (16), this time the balance should be settled by the *kinetic* energy of the particles only.

A calculation of this kind was tried with the data of REEKIE and HUTCHINSON ([14]), which however refer to 2.06 °K. The result is $29.2 \cdot 10^{-16}$ erg/at. for the potential energy, which would require a kinetic energy at absolute zero of $16.7 \cdot 10^{-16}$ erg/at. against a calculated value (eq. (12)) of 14.2. It would of course be difficult to say if this difference is only due to any change in the distribution function between 0 °K and 2.06 °K or if some independent cause of error has contributed to it.

4. – Exchange Integrals. Structure Factor.

We now turn to the calculation of the exchange integral between any two states of type (6).

Its expression can be written

$$I_e(\boldsymbol{K}_1, \boldsymbol{K}_2) = |C_{\boldsymbol{K}_1} C_{\boldsymbol{K}_2}|^2 \iint \sum_m \exp[2\pi i \boldsymbol{K}_1 \cdot \boldsymbol{r}_m] \varphi(|\boldsymbol{\rho}_1 - \boldsymbol{r}_m|) \sum_n \exp[2\pi i \boldsymbol{K}_2 \cdot \boldsymbol{r}_n] \cdot$$

$$\cdot \varphi(|\boldsymbol{\rho}_2 - \boldsymbol{r}_n|) \cdot W(|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|) \sum_{m'} \exp[-2\pi i \boldsymbol{K}_1 \cdot \boldsymbol{r}_{m'}] \varphi(|\boldsymbol{\rho}_2 - \boldsymbol{r}_{m'}|) \cdot$$

$$\cdot \sum_{n'} \exp[-2\pi i \boldsymbol{K}_2 \cdot \boldsymbol{r}_{n'}] \varphi(|\boldsymbol{\rho}_1 - \boldsymbol{r}_{n'}|) \, \mathrm{d}\boldsymbol{\rho}_1 \, \mathrm{d}\boldsymbol{\rho}_2 \, .$$

Two main questions arise in connection with this formula: the one is the proper choice of the domain of integration and the other is whether only a part of the interaction W should be used, since the φ's already take some account of the mutual actions between the atoms. We shall come back to these questions later on.

In performing the integrations indicated in (18), consider any two particular values for $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$: clearly we can safely drop from the sums every

([14]) Ref. ([6]). It is a pleasure to acknowledge the Authors for having communicated privately more details about their distribution function.

term except the one for which $|\boldsymbol{\rho} - \boldsymbol{r}|$ is less than $a/2$ (a internodal distance) ([15]). This allows I_e to be written

$$(19)\quad I_e = |C_{K_1} C_{K_2}|^2 \iint \exp[2\pi i(\boldsymbol{K}_1 - \boldsymbol{K}_2)\cdot(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)] \exp[2\pi i(\boldsymbol{K}_1 - \boldsymbol{K}_2)(\boldsymbol{\delta}_2 - \boldsymbol{\delta}_1)\cdot$$

$$\cdot W(r)|\varphi(|O_m + \boldsymbol{\delta}_1|)\varphi(|O_n + \boldsymbol{\delta}_2|)|^2 \,\mathrm{d}\boldsymbol{\rho}_1 \,\mathrm{d}\boldsymbol{\rho}_2 .$$

Here

$$(19')\quad \begin{cases} \boldsymbol{\delta}_1 = \boldsymbol{\rho}_1 - \boldsymbol{r}_m , \qquad \boldsymbol{\delta}_2 = \boldsymbol{\rho}_2 - \boldsymbol{r}_n & (\delta < a/2) \\ r = |\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2| = |\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2 + \boldsymbol{r}_m - \boldsymbol{r}_n| \end{cases}$$

and O_m, O_n are nodes of the lattice chosen according to the values of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ so that

$$|O_m - \boldsymbol{\rho}_1| , \quad |O_n - \boldsymbol{\rho}_2| < a/2 .$$

It is convenient at this point to perform the integration in a way as similar as possible to the one usually employed for free particles. This can be achieved by first averaging the factors

$$(20)\qquad g = \exp[2\pi i(\boldsymbol{K}_1 - \boldsymbol{K}_2)\cdot(\boldsymbol{\delta}_2 - \boldsymbol{\delta}_1)]$$

and

$$(21)\qquad G = |\varphi(|O_m + \boldsymbol{\delta}_1|)\varphi(|O_n + \boldsymbol{\delta}_2|)|^2 .$$

(Note that G depends on $r = |\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2 + \boldsymbol{r}_m - \boldsymbol{r}_n|$ and must then be averaged by constant r, yielding a function of this variable).

Putting

$$(22)\qquad \boldsymbol{K}_0 = \boldsymbol{K}_1 - \boldsymbol{K}_2 ,$$

g becomes $\exp[2\pi i \boldsymbol{K}_0\cdot\boldsymbol{\delta}_2]\exp[-2\pi i \boldsymbol{K}_0\cdot\boldsymbol{\delta}_1]$. Since $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ vary independently in the interior of the sphere of radius $a/2$, this expression can be easily averaged giving

$$(20')\qquad f(K_0) = \bar{g} = \frac{3^2}{\alpha^6}(\sin\alpha - \alpha\cos\alpha)^2 \qquad \alpha = \pi K_0 a .$$

Fig. 3 is a plot of $\bar{g}$ versus ω_0 defined as

$$(23)\qquad \omega_0 = 2\pi K_0 .$$

([15]) This rule fails at points midway between the nodes, but the number of these points being finite the value of the integral can in no way be affected.

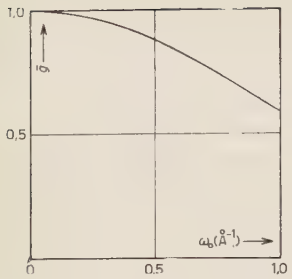

Fig. 3. – The $\bar{g}$ factor (eq.s (20) and (20′)) plotted versus $\omega_0 = 2\pi K_0$.

More laborious, indeed practically impossible to perform rigorously, is the averaging of G (eq. 21)) which results in a function

$$F(r) = (\bar{G})_{r=\text{const.}} \tag{21′}$$

of which Table II and the curve Fig. 4 represent an approximative evaluation

Reverting to the exchange integral (19), it will become

$$I_e = |C_{\boldsymbol{K}_1} C_{\boldsymbol{K}_2}|^2 f(K_0) \iint W(r) \exp[2\pi i \boldsymbol{K}_0 \cdot \boldsymbol{r}] F(r)\, \mathrm{d}\boldsymbol{\rho}_1\, \mathrm{d}\boldsymbol{\rho}_2 .$$

Table II.

r	$F(r)$	r	$F(r)$
0.0	$2.95 \cdot 10^{-3}$	2.6	$3.46 \cdot 10^{-4}$
0.4	2.71 »	3.2	5.12 »
1.0	1.37 »	4.0	4.81 »
1.6	$3.77 \cdot 10^{-4}$	5.0	3.57 »
2.0	1.39 »	∞	4.33 »

(Here $\boldsymbol{r} = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$). This integral can now be transformed by standard methods Introducing a spherical coordinate system having $\boldsymbol{K}_0$ as polar axis and utilizing (23) it can readily be put in the form

$$I_e = |C_{\boldsymbol{K}_1} C_{\boldsymbol{K}_2}|^2 f(K_0) \frac{4\pi V}{\omega_0} \int_{r_0}^{\infty} W(r) F(r)\, r \sin \omega_0 r\, \mathrm{d}r \tag{24}$$

where V is the volume of the liquid and r_0 is to be determined later. (For $C_{\boldsymbol{K}_1}$ and $C_{\boldsymbol{K}_2}$ see eq. (7)). In the limit $K_1 = K_2 = 0$ the above formula yields

$$I_e(0,0) = N^{-2}(1+4\sigma)^{-2} 4\pi V \int_{r_0}^{\infty} W(r)\, F(r)\, r^2\, \mathrm{d}r. \tag{24′}$$

Formula (24) for I_e may be compared with that obtained employing free-particle eigenfunctions instead of eigenfunctions of type (6). The latter reads

$$\frac{4\pi}{V} \frac{1}{\omega_0} \int_{r_0}^{\infty} W(r)\, r \sin \omega_0 r\, \mathrm{d}r. \tag{24″}$$

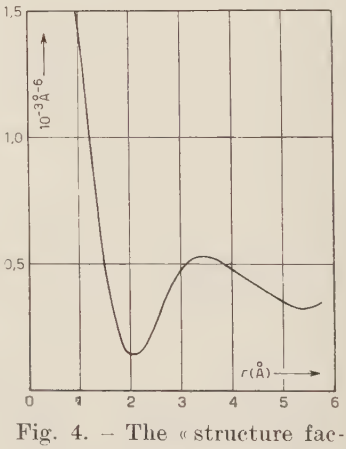

Fig. 4. – The « structure factor » $F(r)$ in the exchange and « Coulomb » integrals.

We see that the main change ([16]) is the appearence in (24) of the « structure factor » $F(r)$. Care must be taken not to confuse this structure factor with any actual distribution function of the distances between the atoms. $F(r)$ is a *fictitious* distribution which has a meaning only within the frame of the perturbation method we are employing here. In fact it implies — as shown by the calculation itself — the free overlapping of the eigenfunctions of the interacting particles, a completely unreal situation.

Along with the exchange integral it is worth while considering the related « Coulomb » i.e. « non-exchange » integral. This will read (same notations as for (18))

$$(25)\quad I_c(\boldsymbol{K}_1, \boldsymbol{K}_2) = |C_{\boldsymbol{K}_1} C_{\boldsymbol{K}_2}|^2 \int\!\!\int \sum_m \exp[2\pi i \boldsymbol{K}_1 \cdot \boldsymbol{r}_m] \varphi(|\boldsymbol{\rho}_1 - \boldsymbol{r}_m|) \cdot$$
$$\cdot \sum_n \exp[2\pi i \boldsymbol{K}_2 \cdot \boldsymbol{r}_n] \varphi(|\boldsymbol{\rho}_2 - \boldsymbol{r}_n|) W(|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|) \sum_{m'} \exp[-2\pi i \boldsymbol{K}_1 \cdot \boldsymbol{r}_{m'}] \cdot$$
$$\cdot \varphi(|\boldsymbol{\rho}_1 - \boldsymbol{r}_{m'}|) \sum_{n'} \exp[-2\pi i \boldsymbol{K}_2 \cdot \boldsymbol{r}_{n'}] \varphi(|\boldsymbol{\rho}_2 - \boldsymbol{r}_{n'}|) \, \mathrm{d}\boldsymbol{\rho}_1 \, \mathrm{d}\boldsymbol{\rho}_2 \, .$$

This expression can be treated exactly as the analogous for I_e. The important difference is that now the dependence on $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ drops from the integrand, so that the final result is simply

$$(26)\qquad I_c = |C_{\boldsymbol{K}_1} C_{\boldsymbol{K}_2}|^2 4\pi V \int_{r_0}^{\infty} W(r) F(r) r^2 \, \mathrm{d}r \, .$$

It will be noted that in the limit $K_1 = K_2 = 0$, this expression coincides with the corresponding one for I_e. Thus

$$(27)\qquad I_e(0, 0) = I_c(0, 0) \, .$$

5. – Empirical about the Value of Exchange Integrals. The Energy Absorbed per Evaporated Particle.

If we were dealing with simple point particles, there would be no doubt that the lower limit r_0 in the integrals (24) and (26) should be put zero. Here however we are trying to extend the formalism suitable for point particles to composite systems which loose even their definitedness when coming sufficiently close together. This procedure is only to be justified on grounds of simplicity and just to render it possible we have implicitly introduced a « cut » by writing r_0 instead of zero.

([16]) The difference in the dependence on V disappears taking (7) into account and noting that N and V are proportional.

To get an idea about how r_0 should be determined, let us go back to experience and investigate the energy absorbed per evaporated particle. We shall call this quantity $\mathcal{E}$ and define it as

$$\mathcal{E} = \frac{1}{N_n(T)} \int_0^T C \, \mathrm{d}T \,. \tag{27}$$

Here C is the (molal) specific heat and N_n is the number (per mole) of independent elements belonging to excited states. The latter quantity can be deduced from the experimental data on the density ϱ_n of the normal fluid provided we can make some appropriate assumption about the mass of the elements concerned. Two assumptions will be tested here: A) the elements are phonons, B) the elements are particles having a mass of the order of m_{He}. It will turn out very clearly that each one of these assumptions has its own range of validity.

The testing of the second assumption is immediate. The first one, on the contrary, need some preliminary investigation. Let us therefore examine the question of phonons in a little detail.

Phonons are characterized by an energy cp where c is the velocity of ordinary sound and p is the momentum. For more generality, we shall write

$$E = \Delta + cp \,, \tag{28}$$

with Δ a constant.

Standard methods ([17]) lead for a saturated Bose-Einstein perfect gas with an energy spectrum (28) to following formula for N_n

$$N_n = \frac{4\pi V}{h^3} \int_0^\infty \frac{p^2 \, \mathrm{d}p}{\exp\left[(\Delta/kT) + (cp/kT)\right] - 1} = \tag{29}$$

$$= \frac{4\pi V}{h^3} \left(\frac{kT}{c}\right)^3 2 \left[\frac{\exp\left[-\Delta/kT\right]}{1^3} + \frac{\exp\left[-2\Delta/kT\right]}{2^3} + \ldots\right]$$

and for $\bar{p}$ (average momentum)

$$\bar{p} = \frac{1}{N_n} \frac{4\pi V}{h^3} \int_0^\infty \frac{p^3 \, \mathrm{d}p}{\exp\left[(\Delta/kT) + (cp/kT)\right]} =$$

$$= \frac{1}{N_n} \frac{4\pi V}{h^3} \left(\frac{kT}{c}\right)^4 3! \left[\frac{\exp\left[-\Delta/kT\right]}{1^4} + \frac{\exp\left[-2\Delta/kT\right]}{2^4} + \ldots\right].$$

([17]) See for instance R. B. Dingle: *Phil. Mag. Suppl.*, **1**, 111 (1952).

We shall distinguish two limiting cases: i) $\Delta = 0$; ii) Δ sufficiently large in order that only the first term in each of the above series need be retained. One finds

$$\text{(30)} \qquad \begin{cases} N_n = 2.404 \dfrac{4\pi V}{h^3}\left(\dfrac{kT}{c}\right)^3 \\ \bar{p} = 2{,}70 \dfrac{kT}{c} \end{cases} \qquad \text{for case i)}$$

$$\text{(30')} \qquad \begin{cases} N_n = 2 \dfrac{4\pi V}{h^3}\left(\dfrac{kT}{c}\right)^3 \exp\left[-\dfrac{\Delta}{kT}\right] \\ \bar{p} = 3 \dfrac{kT}{c} \end{cases} \qquad \text{for case ii)}$$

Assuming phonons to travel with velocity c, their average mass can be deduced from $\bar{p}$, by writing

$$\text{(31)} \qquad \bar{\mu}_{\text{ph.}} c = \bar{p}$$

with the results

$$\text{(31')} \qquad \text{i)} \quad \bar{\mu}_{\text{ph.}} = 2.70 \frac{kT}{c^2}; \qquad \text{ii)} \quad \bar{\mu}_{\text{ph.}} = 3 \frac{kT}{c^2}.$$

Assuming moreover that we can write for the density of the normal fluid

$$\text{(32)} \qquad \varrho_n = \frac{N_n}{V} \bar{\mu}_{\text{ph.}}$$

we find for ϱ_n from (30), (30'), (31'), (32)

$$\text{(33)} \qquad \begin{cases} \text{case i)} \quad \varrho_n = 6.491 \dfrac{4\pi}{h^3} \dfrac{1}{c}\left(\dfrac{kT}{c}\right)^4 \\ \text{case ii)} \quad \varrho_n = 6.000 \dfrac{4\pi}{h^3} \dfrac{1}{c}\left(\dfrac{kT}{c}\right)^4 \exp\left[-\Delta/kT\right]. \end{cases}$$

Indeed, the results of DE KLERK, HUDSON and PELLAM [18] giving practically a law in T^4 for $T \to 0$ rule out case ii).

We are now in position to try an evaluation of $\mathcal{E}$ eq. (27). In the phonon assumption, from eq.'s (27), (32), (31' i))

$$\text{(34.A)} \qquad \mathcal{E} = \frac{2.7}{Vc^2} \frac{kT}{\varrho_n} \int_0^T C \, dT. \qquad \text{(phonons).}$$

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[18] D. DE KLERK, R. P. HUDSON and J. R. PELLAM: *Phys. Rev.*, **93**, 28 (1954).

In the « atomic » assumption

$$\mathcal{E} = \frac{m}{V\varrho_n}\int_0^T C\,\mathrm{d}T\,. \qquad \text{(particles of atomic mass).} \tag{34.B}$$

The data employed for C are those of ref. ([13]). For ϱ_n the measurements of ref. ([18]) have been utilized in the region below 1 °K and those of MAURER and HERLIN ([19]) and HOLLIS-HALLETT ([20]) above the same temperature. The « atomic mass » employed is m_{He} itself.

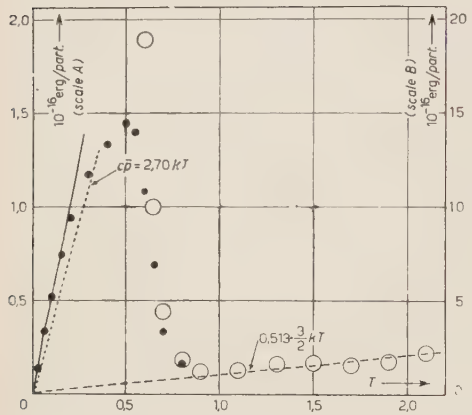

Fig. 5. – Energy absorbed per exciton, as a function of T. *Full circles*: phonon hypothesis; dotted line: average phonon energy (scale A). *Open circles*: atomic mass hypothesis; broken line: average energy for saturated perfect monoatomic Bose-Einstein gas (scale B). Data for ϱ_n: ref. ([18]), ([19]). ([20]). Data for specific heat: ref. ([13]).

The results are reported in Fig. 5.

The small, full circles are obtained from eq. (34 A)), the open ones from eq. (34 B)). (Note difference in scales by a factor 10).

We see that to obtain reasonable results the phonon hypothesis is to be used below, say, 0.3 °K, while the atomic hypothesis must be employed above about 0.8 °K. In the intermediate region neither give good results. The rapid falling of the full circles with increasing temperature should be explained most naturally by the onset of excitons having far greater mass than phonons, thus leading to overevaluate the number of these. Conversely the rise of the open circles by decreasing temperature should mark the disappearence of « heavy » excitons and the onset of « light » ones.

Except at the lowest temperatures, the diagram must be considered as very « qualitative », but its answer is so clear that it appears not unsafe to draw from it following conclusions:

a) Between 0.3 and 0.8 °K there takes place a rapid change in the character of the thermal motion, but

b) there is no hint af any large excitation energy.

([19]) R. MAURER and M. HERLIN: *Phys. Rev.*, **76**, 948 (1949).
([20]) A. C. HOLLIS-HALLETT: *Proc. Roy. Soc. Lond.*, A **210**, 404 (1952).

Because of uncertainties that may well be of the order of 100% in the central region of the diagram, statement *b*) is not intended to mean there is no excitation energy at all, but that its value cannot be of the order of 10 to $15 \cdot 10^{-16}$ erg/at. as often postulated in this connection.

Perhaps we could venture as much as to add that

c) any excitation energy should *decrease* with increasing temperature, in such a way as to leave practically no trace when « evaporation » has become total, at the λ-point. Exchange forces can offer a possibility in this sense ([21]).

The energy per evaporated particle $\mathcal{E}$ is closely related to exchange forces. Indeed at absolute zero there can be no exchange, since all the particles are in the same state. Exchange energy can be present as soon as there are particles occupying excited states. For Bose-Einstein particles this energy has the same sign as the exchange integral: we can therefore infer that, 1) this integral must be non-negative, as otherwise excited states would lie lower than the fundamental one, 2) exchange energy should therefore act as an excitation energy.

Near absolute zero we should expect K_1, K_2 (see Sect. **4**) to tend to zero: we have then a limiting value $I_c(0, 0)$ for the exchange integral and this coincides (eq. (27)) with that of the « non-exchange » integral $I_c(0, 0)$. It might appear surprising at first sight that $I_c(0, 0)$, too, should be non-negative, since one expects it to represent the mutual potential energy of two particles and this must of course be negative. Indeed $I_c(0, 0)$ represents only the potential energy corresponding to a fictitious state of affairs, in which the particle distributions overlap freely, thus giving an excess repulsive energy which would have to be subtracted from I_c to get the correct potential energy.

As a matter of fact, the empirical data we have just discussed about $\mathcal{E}$ seem to indicate that the exchange integrals are always small, corresponding to an average energy per particle not larger than a few units in 10^{-16} erg/at..

6. – Value of r_0. Dependence of the Exchange Integrals on K_0. The Effective Mass Again.

It is possible to choose for r_0 a value which renders the exchange integral as small as we please, thus satisfying the condition at the end of the previous section. This happens because of compensation between the contributions from the repulsive and attractive terms in W.

([21]) A decreasing excitation energy, due to exchange forces, was put forward first in ref. ([1]).

Fig. 6 shows a plot of the absolute values of these contributions versus r_0 for the case of $I_e(0, 0)$. For convenience, the ordinates have been multiplied by N. (Note that

$$\mathcal{E}_e = NI_e \tag{35}$$

is the exchange energy per evaporated atom, in case the evaporated atoms are interacting in the same manner with every other, as is practically the case when their concentration is small).

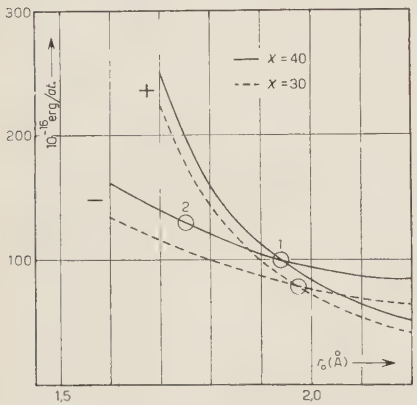

Fig. 6. – Absolute values for the positive and negative terms in the exchange integral between states of momentum $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$; in the limit $K_1, K_2 \to 0$, plotted against the value of the lower limit r_0.

The point labelled 2 on Fig. 6 shows where compensation would occur if the repulsive term in W were reduced by a factor $\frac{2}{3}$. (See considerations at the beginning of Sect. 4). Thus we see that in any case r_0 should not be far from 2 Å.

Formula (24) shows that I_e is essentially a function $I_e(K_0) = I_e(|\boldsymbol{K}_2 - \boldsymbol{K}_1|)$, the separate dependence on $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ coming only from the unimportant factor $|C_{\boldsymbol{K}_1} C_{\boldsymbol{K}_2}|^2$. Now it turns out that if we choose r_0 so as to have a value zero (or practically such) for $I_e(0)$, the same does not happen for $K_0 > 0$, but instead

$$I_e(K_0) > 0\,, \qquad (K_0 > 0).$$

This is important because the expression for the total exchange energy of the liquid (to be derived in the next Section) includes terms in $I_e(K_0)$ with K_0 an increasing function of T.

Of course one could at any temperature determine r_0 so that the exchange energy — and along with it any excitation energy — stay small. This however would not be very satisfactory, as we should expect the value of r_0 to be fixed by some internal feature of the exchange mechanism and thus be independent of any state parameter ([22]).

Assuming then r_0 to be constant, a very interesting possibility arises from the presence of the structure factor $F(r)$ in the exchange integral, and this is that the local eigenfunctions φ should adjust themselves so as to modify $F(r)$ and thus keep the average value of the integrals (as well as the exchange

([22]) The eigenfunctions themselves should conform to the value of r_0 and not viceversa. The appearence of the contrary shown by the above evaluation of r_0 is however imposed by the semiempirical approach adopted here.

energy) from increasing too much. This constitutes the suggested root of the negative expansion, on which we shall come back again (Sect. **7**).

The dependence of the exchange integral on K_0 is due to the presence in the integrand of the function (eq. (24))

$$u(\omega_0, r) = \frac{\sin \omega_0 r}{\omega_0},$$

which does not appear in the corresponding non-exchange integral, so that the dependence in question *is absent* in the case of the latter.

To study I_e as a function of K_0 would need the numerical evaluation of I_e for several values of K_0. A more convenient approximative method requiring only two numerical integrations (in addition of those already performed) is the following one.

The integral to be studied can be written

$$(36) \qquad \int_{r_0}^{\infty} WFr \frac{\sin \omega_0 r}{\omega_0} \mathrm{d}r = \int_{r_0}^{\infty} W_+ Fr \frac{\sin \omega_0 r}{\omega_0} \mathrm{d}r + \int_{r_0}^{\infty} W_- Fr \frac{\sin \omega_0 r}{\omega_0} \mathrm{d}r$$

where W_+ and W_- are the positive and negative term in (1). Introducing appropriate averages $\bar{r}$ of r, each term in the right hand side of the above equation can be approached, as far as its dependence on ω_0 is concerned, by a function

$$A \frac{\sin \bar{r}\omega_0}{\omega_0}.$$

The averages $\bar{r}$, denoted r_+ and r_-, may be computed in the limit $\omega_0 \to 0$ (when the integrals are simpler) from the formula [23]

$$(37) \qquad r_{\pm} \int_{r_0}^{\infty} W_{\pm} Fr^2 \mathrm{d}r = \int_{r_0}^{\infty} W_{\pm} Fr^3 \mathrm{d}r$$

and of course r_- turns out to be larger than r_+, as the attractive forces decrease less rapidly with distance. We shall put for convenience

$$(37') \qquad \varepsilon = r_- - r_+ .$$

---

[23] This formula characterizes r_+ and r_- as « centers of gravity » of the positive and negative contributions to

$$\int_{r_0}^{\infty} WFr^2 \mathrm{d}r .$$

Numerical values are collected in Table III:

TABLE III.

r_0	$\chi = 40 \cdot 10^{-16}$ erg/Å			$\chi = 30 \cdot 10^{-16}$ erg/Å		
	r_+	r_-	ε	r_+	r_-	ε
2.2	2.60	3.52	0.93	2.70	3.76	1.06
2.1	2.49	3.46	0.97	2.60	3.67	1.07
2.0	2.39	3.37	0.98	2.46	3.54	1.08
1.9	2.32	3.25	0.93	2.32	3.43	1.11
1.8	2.21	3.07	0.86	2.17	3.27	1.10

and show that ε is rather insensitive to the choice of the quantities r_0 and χ. The value $\varepsilon = 1$ Å will be chosen for simplicity.

Reverting to eq. (36), we replace it by

$$\int_{r_0}^{\infty} WFr \frac{\sin \omega_0 r}{\omega_0} \, \mathrm{d}r = \alpha \frac{\sin \omega_0 r_+}{\omega_0} - \beta \frac{\sin \omega_0 r_-}{\omega_0} . \tag{38}$$

Assuming this integral to be exactly zero in the limit $\omega_0 \to 0$, one finds

$$\beta = \alpha \frac{r_+}{r_+ + \varepsilon} = \alpha \left(1 - \frac{\varepsilon}{r_+} + \frac{\varepsilon^2}{r_+^2} - \ldots \right) \tag{38'}$$

as well as

$$\alpha r_+ = \int_{r_0}^{\infty} W_+ F r^2 \, \mathrm{d}r , \qquad \beta r_- = \int_{r_0}^{\infty} W_- F r^2 \, \mathrm{d}r \tag{38''}$$

by means of which the numerical constants may be computed. Values adopted are [24]

$$r_+ = 2.36 \text{ Å} , \quad \varepsilon = 1 \text{ Å} , \quad \alpha = 0.086 , \quad \beta = 0.061 . \tag{38'''}$$

Substitution from (37') and (38') in (38) and development in series of powers of ε / r_+ to the 1st order of this quantity gives

$$\int_{r_0}^{\infty} WFr \frac{\sin \omega_0 r}{\omega_0} \, \mathrm{d}r = \alpha \varepsilon \left(\frac{\sin \omega_0 r_+}{\omega_0 r_+} - \cos \omega_0 r_+ \right) , \tag{39}$$

which means, practically, a quadratic increase of the integral with ω_0.

[24] These values, as indeed any other given so far, are only tentative and must not be given any too precise meaning.

It is possible to find a simpler expression for the product of the above formula by f, eq. (20), which is also a function of ω_0. Numerical computation shows that

$$f(\omega_0)\left(\frac{\sin\omega_0 r_+}{\omega_0 r_+}-\cos\omega_0 r_+\right)=1.9\,\omega_0^2-1.7\,\omega_0^4 \tag{40}$$

holds within 2.5 % in the range of interest ($0<\omega_0<$ about 0.7). From this formula, eq.'s (24), (7) and the numerical values (38‴), the following expression is obtained for I_e

$$I_e=\frac{4\pi V}{N^2}\,\frac{0.086(1.9\,\omega_0^2-1.7\,\omega_0^4)}{(1+4\sigma\cos\pi K_1 a)(1+4\sigma\cos\pi K_2 a)} \tag{40'}$$

With energies in 10^{-16} erg and lenghts in Å, and with the known value of the atomic volume $V=27.5$ cm^3, the same equation yields

$$NI_e=\frac{49.1(1.9\,\omega_0^2-1.7\,\omega_0^4)}{(1+0.0772\cos 5.02K_1)(1+0.0772\cos 5.02K_2)}\,. \tag{40''}$$

Fig. 7 is a plot of the ω_0-dependent part of NI_e. The scale is as in Fig. 6.

In case the integral (36) is not exactly zero for $\omega_0\to 0$ but has a small value (small compared with (38″)), then one finds without difficulty, instead of (39)

$$\int_{r^0}^{\infty} WFr\,\frac{\sin\omega_0 r}{\omega_0}\,\mathrm{d}r=\alpha'\varepsilon\left(\frac{\sin\omega_0 r_+}{\omega_0 r_+}-\cos\omega_0 r_+\right)+\Delta\,\frac{\sin\omega_0 r_+}{\omega_0 r_+}\,, \tag{39'}$$

with

$$\alpha'=\alpha-\frac{\Delta}{r_+}\,. \tag{39''}$$

This is probably the real case, since otherwise one could not see why quadratic excitations should disappear below about 0.5 °K. For the sake of simplicity we shall use however the simpler formula (39).

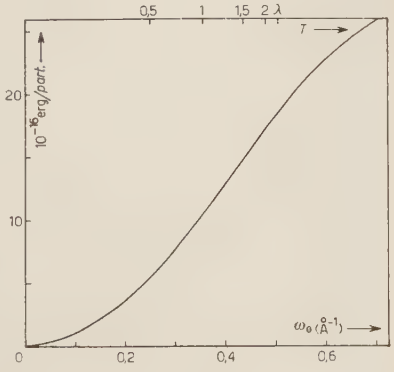

Fig. 7. – The value of the exchange integral (in the assumption that it is zero in the limit $K_1,\ K_2\to 0$) as a function of $\omega_J=2\pi|\boldsymbol{K}_1-\boldsymbol{K}_2|$. Upper abscissa scale: temperatures according to eq. (43).

The fact that I_e is a function of ω_0 (and in particular a practically quadratic one) has an important bearing on the apparent mass. Consider for instance the simplest case $R_n \ll 1$. Then, to « evaporate » a particle we must give it a kinetic energy $(1/2\mu)(h^2\omega_0^2/4\pi^2)$ plus the « non-neutralized » fraction η of the exchange energy $\eta N I_e$ (eq. (35)). In total (eq. 40″), units as usual)

$$\frac{1}{2\mu}\frac{h^2\omega_0^2}{4\pi^2} + 80.4\,\eta\omega_0^2 \qquad (R_n \ll 1)$$

and putting this equal to $(1/2\mu^*)(h^2\omega_0^2/4\pi^2)$ we get, measuring the μ's in units of m_{He}

$$\frac{1}{\mu_R^*} = \frac{1}{\mu_R} + 9.62\,\eta. \qquad (R_n \ll 1). \tag{41}$$

This equation shows that to determine the effective mass, a thorough discussion of the « exchange equilibrium » (on which the value of η depends) is required. Since μ_R is about 2 (eq. (15)), we should have $\mu_R^* \approx 1$ with $\eta = 10^{-1}$. Here lies perhaps the ground of the fact that we had to use m_{He} in computing the data reported in Fig. 5 ([25]).

7. – Negative Expansion and Exchange Energy.

We shall now put to the test the proposed explanation of the negative expansion as a check of the model put forward in the previous sections.

A more or less rigorous treatment would be fairly intricate. Here we shall confine ourselves to compare — so to speak — the effect to its presumed cause, putting the density increase above 1 °K in relation with $E_e(T)$, by which notation we shall indicate the ratio to N of the total exchange energy, or — better — *what this quantity would be* in the absence of any counteracting mechanism, all other features remaining the same.

With the customary simplifications, E_e is given by

$$NE_e = \tfrac{1}{2}\sum_x \sum_{y \neq x} n_x n_y I_{xy} = n_0 \sum_{x=1}^{\infty} n_x I_{x0} + \tfrac{1}{2}\sum_{x=1}^{\infty}\sum_{y=1}^{\infty} n_x n_y I_{xy}, \tag{41}$$

([25]) It must however be noted that it is not certain that the masses appropriate to interpret the types of measurement on which diagram 5 is based should be the same as here defined.

Added in proof: The above considerations are incomplete also in other respects. Indeed, as soon as the local eigenfunctions φ are modified (and with them the structure factor F) the value of μ_R is no longer the same (eq. 14) and (8), (10)). Moreover, there is a (negative) contribution to the energy from the « Coulomb » integral, which is no longer zero. All that deserves a deeper analysis.

where x and y label the states, n_x and n_y are the corresponding occupation numbers and I_{xy} has been written for I_e relative to the states x, y. (Account has been taken that only n_0 is a « macroscopic » number).

Introducing average values, the above equation can be written

$$NE_e = n_0\left(\sum_{x=1}^{\infty} n_x\right)\overline{I_{x0}} + \tfrac{1}{2}\left(\sum_{x=1}^{\infty} n_x\right)\overline{I_{xy}} .$$

Putting

$$\frac{1}{N}\sum_{x=1}^{\infty} n_x = R_n \tag{41'}$$

for the ratio of the number of particles occupying « normal fluid » states [26] to the total number of particles N, and, as a consequence,

$$\frac{n_0}{N} = 1 - R_n , \tag{41''}$$

the above expression for E_e becomes

$$E_e = N\left[\, R_n(1 - R_n)\,\overline{I_{x0}} + \tfrac{1}{2}R^2\overline{I_{xy}}\,\right]. \tag{42}$$

Above 1 °K we may take the data on ϱ_n/ϱ as at least approached values for R_n. Thus it remains to determine $\overline{I_{x0}}$ and $\overline{I_{xy}}$ to know E_e as a function of T. As these integrals are known as functions of ω_0, this means to know how ω_0 varies with T.

No exact answer can be given to this problem without a complete analysis of the exchange equilibrium. The simplest a priori assumption is suggested by diagram Fig. 5 and is to adopt a « perfect gas treatment » taking

$$\frac{1}{2\mu^*}\,h^2K^2 = \frac{1}{2\mu^*}\,\frac{\overline{\omega^2}}{4\pi^2} \approx 0.5\,\frac{3}{2}\,kT .$$

With μ^* of the order m_{He} that gives

$$\overline{\omega^2} = 0.127\,T\ \text{Å}^{-2} . \tag{43}$$

We shall make use of this admittedly very tentative formula. It gives the

[26] As such, that is as excited states, only those of type (6) are considered here, since we are not interested in temperatures below 0.8 °K, where phonons are important.

proper value to be substituted for ω_0^2 in (40″) to get I_{x0}, while for I_{xy} this value has to be doubled, as elementary considerations readily show.

In this way E_e has been computed from (42) using the data of ref. ([18]), ([19]), ([20]) for ϱ_n/ϱ. The results are reported in Table IV (last column). They should

TABLE IV.

T	$\mathcal{E}_e$	E_e	T	$\mathcal{E}_e$	E_e
0.8	7	0.007	2.0	13.5	7.82
1.0	9.0	0.065	2.1	12.7	8.95
1.2	10.6	0.296	2.15	12.5	10.60
1.4	12.0	0.719	2.19	11.72	11.72
1.6	13.1	2.14	2.4	11.66	11.66
1.7	15.5	3.24	2.6	11.40	11.40
1.8	13.7	4.53	2.8	10.98	10.98
1.9	13.7	6.04			

be compared with the density increase under negligible pressure (practically also of the order of 1 atm), since nowhere a pressure has been taken into

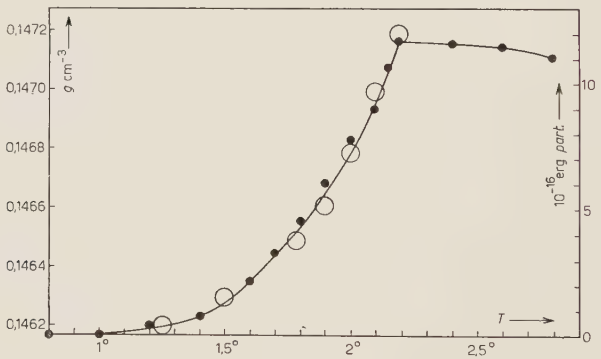

Fig. 8. - In this diagram, the data of ref. ([27]) for the density of He II under $p=1$ atm (open circles, left-hand scale) are plotted versus T together with E_e of Table IV (full circles, curve; right-hand scale). The latter quantity is the value which the average exchange energy per atom would take if – all other things being the same – the local eigenfunctions φ kept unmodified.

account and, moreover, the numerical data (as for instance $a = 3.2$ Å) refer to this case.

In Fig. 8 the data of KEESOM and KEESOM ([27]) for $p = 1$ atm and the

([27]) W. H. KEESOM: *Helium* (Amsterdam, 1942), p. 240.

above results (E_e) are plotted together ([28]) and agreement may be seen to be good.

Beyond the λ-point the exchange energy decreases slowly, thus contributing to the normal expansion.

The values of $\mathcal{E}_e$ in Table IV, i.e. the ratios of E_e to ϱ_n/ϱ are also interesting, as they show but little variation in comparison with E_e. This means that the density increase (which parallels E_e) is governed by the number of interacting particles rather than by the importance of their interaction. The simplest interpretation of this fact is that there are « shrinked » eigenfunctions (i.e. built by combining « shrinked » φ's) mixed in increasing proportion to normal ones ([29]). This is particularly understandable in the limit $R_n \ll 1$. Clearly, since in that case the exchange is essentially between the few « evaporated » atoms and those in the ground state, forming the bulk of the liquid, it is more « economical » to shrink only the eigenfunctions of the former ones.

It is noteworthy that this can be done without impairing orthogonality. In fact, the orthogonality conditions for a system of functions of the type

$$\bar{\psi}_{\boldsymbol{K}} = \sum_n C_{\boldsymbol{K}} \exp\left[2\pi i \boldsymbol{K}\cdot\boldsymbol{r}_n\right] \varphi_{\boldsymbol{K}}(|\boldsymbol{\rho} - \boldsymbol{r}_n|) \tag{44}$$

are the same as for φ independent of $\boldsymbol{K}$ ([30]).

The smallness of the density effect should be explained in this scheme by the fact that only the outer fringe, so to speak, of the local eigenfunction φ, around and beyond, say, $\varrho = 1.6$ Å needs be decreased. The volume of the liquid — on the other hand — is likely to depend on some « breadth » of the eigenfunction, which should be but little affected.

8. – Conclusion.

The nearly quantitative agreement between density increase and exchange energy shown by the plot in Fig. 8 may be partly due to compensation of

([28]) It might be argued whether E_e should not be compared with the difference between the actual volume and that expected by extrapolating the « normal » law of expansion of He I below the λ-point. This however seems not to be right, since the value of the exchange integrals will depend on the *actual* volume and there can be little compromise between the normal expansion which involves energies of the order of $k\Delta T$ and the « exchange contraction » which involves energies nearly 10 times as large (in the interval between 1 °K and the λ-point).

([29]) The shrinking, by reducing overlapping should obviously keep the exchange energy low.

([30]) They amount to put

$$\sum_n \exp\left[2\pi i(\boldsymbol{K} - \boldsymbol{K}')\cdot\boldsymbol{r}_n\right] = 0\,,$$

which brings the well known limitations for the $\boldsymbol{K}$'s.

errors. It appears however sufficient to indicate that exchange forces play indeed a role in the behaviour of He II. This role deserves a more complete analysis and is probably decisive for many aspects of the problem, in particular, excitation energies, apparent masses, « evaporation » law.

RIASSUNTO

Partendo da un particolare « metodo cellulare » si costruiscono autofunzioni (di una particella) che resultano appropriate per descrivere quegli stati eccitati dell'elio liquido la cui energia è funzione quadratica della quantità di moto. Si studiano poi gli integrali di scambio e « coulombiano » per le dette autofunzioni. Indipendentemente, una analisi dei dati sperimentali mostra che l'energia di scambio (fra 0 °K e il punto λ) deve essere piccola. Siccome si può mostrare che gli integrali di scambio non possono avere un valore fisso, ma devono crescere con la temperatura, ne consegue che deve esistere un meccanismo (contrazione delle autofunzioni) che ne riduce il valore. Si mostra che questo meccanismo dev'essere lo stesso che produce la caratteristica « dilatazione negativa » fra circa 1 °K e la temperatura di transizione. Altri punti presi in considerazione riguardano masse apparenti, energie di eccitazione, vapore saturo di fononi.

Sulla frequenza degli elementi chimici.

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(ricevuto il 27 Agosto 1954)

Riassunto. — Si applicano le formule di termodinamica statistica, ottenute tenendo conto dell'energia relativistica delle particelle e della creazione delle coppie, allo studio della distribuzione degli elementi chimici nell'universo, ammettendo che gli elementi si siano formati in condizioni sufficientemente vicine a quelle di equilibrio termodinamico, in modo da giustificare l'assunzione della validità approssimata di tali formole. Partendo dai tentativi di spiegare con la teoria termodinamica la distribuzione osservata degli elementi, si trova che un accordo con le osservazioni si ottiene solo per densità $\varrho \gtrsim 10^{13}$ g cm^{-3}; inoltre si trova un limite superiore, $m \lesssim 10^{35}$ g, per la massa totale in accordo coi dati astrofisici sulle masse stellari.

1. – Riassumiamo i risultati ottenuti precedentemente da G. Wataghin [1-3] (risultati simili, con metodi termodinamici, sono pure stati ottenuti da vari autori [4]). Ammettiamo che le formule che useremo siano ancora applicabili nelle condizioni che prevalevano nell'epoca in cui si formarono i nuclei.

Si consideri un insieme di particelle (nucleoni, elettroni, mesoni, nuclei neutrini) e di radiazione in condizioni di equilibrio termodinamico a temperatura $kT \sim 10^7$ eV ($T \sim 10^{11}$ °K) e densità $\varrho \gtrsim 10^{12}$ g·cm^{-3}.

Siano N_s, N_{es}, $n_{\mu s}$, $n_{\pi s}$, $n_{\nu s}$, $n_{a\nu s}$, n_{ZAs} i numeri per cm^3 di fotoni, elettroni, mesoni μ, mesoni π, neutrini, antineutrini e nuclei di carica Z e numero di massa A, aventi momenti compresi fra p_s e $p_s+\mathrm{d}p_s$. Sia φ il potenziale del campo gravitazionale esterno.

(1) G. Wataghin: *Phil. Mag.*, **17**, 910 (1934).
(2) G. Wataghin: *Compt. Rend.*, **203**, 909 (1935).
(3) G. Wataghin: *Suppl. Nuovo Cimento*, **9**, 241 (1949).
(4) Per la bibliografia vedi R. A. Alpher e R. C. Hermann: *Rev. Mod. Phys.*, **22**, 153 (1950).

Consideriamo in particolare le seguenti reazioni

$$(1)\qquad \begin{cases} \mathrm{p} \rightleftarrows \mathrm{n} + \mathrm{e}^+ + \nu^* & \mathrm{n} \rightleftarrows \mathrm{p} + \mathrm{e}^- + \nu \\ \pi^+ \rightleftarrows \mu^+ + \nu^* & \pi^- \rightleftarrows \mu^- + \nu \\ \mu^+ \rightleftarrows \mathrm{e}^+ + \nu + \nu^* & \mu^- \rightleftarrows \mathrm{e}^- + \nu + \nu^* \end{cases}$$

processi $\pm\beta$ ed anche reazioni nucleari con produzione e assorbimento di pioni e di altri mesoni. Valgono le seguenti leggi di conservazione delle quali diremo il significato:

$$(2)\qquad \sum_{ZAs} Z n_{ZAs} + \sum_s [n_{e^+s} + n_{\mu^+s} + n_{\pi^+s} - n_{e^-s} - n_{\mu^-s} - n_{\pi^-s}] = n\,,$$

$$(3)\qquad \sum_{ZAs} n_{ZAs} \cdot E_{ZAs} \cdot \left(1 + \frac{\varphi}{c^2}\right) +$$

$$+ \left(1 + \frac{\varphi}{c^2}\right) \sum_s [N_s h\nu_s + n_{e^-s} E_{e^-s} + \ldots + n_{\alpha\nu s} E_{\alpha\nu s}] + \mathcal{E} = E\,,$$

dove $\mathcal{E}$ indica l'energia di interazione, che sarà trascurata nel seguito, ed $E_{ZAs} = m_{ZAs} \cdot c^2$, ecc.,

$$(4)\qquad \sum A \cdot n_{ZAs} = N\,,$$

$$(5)\qquad \sum Z \cdot n_{ZAs} - \sum (A - Z) \cdot n_{ZAs} +$$

$$+ \sum [n_{e^+s} - n_{e^-s} + n_{\alpha\nu s} - n_{\nu s} + n_{\mu^+s} - n_{\mu^-s} + 2(n_{\pi^+s} - n_{\pi^-s})] = S\,.$$

Mentre nell'usuale termodinamica si tien conto solo della conservazione del numero totale delle particelle e dell'energia cinetica, qui con la (4) si è imposta la conservazione del numero totale di nucleoni (si ritiene quindi che gli eventi di produzione di coppie nucleone-antinucleone abbiano frequenza trascurabile) (*), con la (3) si impone la conservazione dell'energia totale, tenendo conto delle trasformazioni reciproche di masse in radiazione, la (2) esprime la conservazione della carica, ed infine la (5) è una relazione di conservazione introdotta da G. Wataghin [3] che deve essere soddisfatta quando avvenga una reazione nucleare del tipo (1), come si verifica facilmente. In un modo simile si potrebbe

(*) È ovvio che nel caso dell'esistenza di antiprotoni e di antineutroni, al primo membro della (4) dovrebbe comparire la somma delle differenze fra i numeri di nucleoni e di antinucleoni: $\sum [n_p - n_{ap} + (n_\nu - n_{a\nu})]$.

considerare anche la creazione di pioni neutri, ma ciò non è necessario per i nostri scopi.

Seguendo i metodi usuali di Boltzmann-Gibbs, le condizioni di equilibrio si trovano determinando il massimo di

$$f = \log W - \alpha n + \beta E + \gamma N - \delta S ,$$

dove

$$W = \prod_s \frac{(g_s + N_s - 1)!}{(g_s - 1)!\, N_s!} \cdot \frac{g_s!}{n_{es}!\,(g_s - n_{es})!} \cdots ,$$

e

$$g_s = V \cdot 8\pi h^{-3} p_s^2 \, \mathrm{d}p_s , \tag{6}$$

ove il volume V è unitario. Si ottengono così i numeri di particelle di ogni tipo relativi ad una celletta dello spazio delle fasi [3]. A noi interessa solo il numero di nuclei per cm^3 delle diverse specie:

$$n_{ZAs} = \frac{g_s}{\exp[\alpha^* Z - \gamma^* A + \beta E^*] \pm 1} , \tag{7}$$

dove:

$$\alpha^* = \alpha + 2\delta , \qquad {}^*\gamma = \gamma + \delta , \qquad E^* = E\left(1 + \frac{\varphi}{c^2}\right),$$

$$\beta = \frac{1}{kT} . \tag{8}$$

Con una integrazione sui momenti, trascurando ± 1 nel denominatore della (7), e sommando sugli stati eccitati si ottiene:

$$n_{ZA} = 2\left[\frac{2\pi m_{ZA}}{h^2} \cdot \frac{kT}{(1 + \varphi/c^2)}\right]^{\frac{3}{2}} \cdot \Gamma_{ZA} \cdot \exp[-\alpha^* Z + \gamma^* A - \beta m_{ZA} c^2 (1 + \varphi/c^2)] , \tag{9}$$

dove

$$\Gamma_{ZA} = \sum_i f_i \exp[-\beta \cdot \Delta E_i (1 + \varphi/c^2)]$$

è la somma sugli stati eccitati aventi un eccesso di energia ΔE_i ed un peso statistico f_i. Per la validità della (9) si suppone che lo stato dei nuclei sia uno stato non degenere. Ci occuperemo più avanti di una formula per i neutroni, in cui si tien conto della degenerazione.

2. – Poichè nella (9) la temperatura T compare col fattore $1/(1+\varphi/c^2)$ sia a coefficiente che ad esponente (è $\beta = 1/kT$), d'ora innanzi converremo di scri-

vere T al posto di $T/(1+\varphi/c^2)$. Perciò la T che useremo rappresenterà una temperatura fittizia che risulterà inferiore a quella vera, perchè $1+\varphi/c^2<1$, essendo il potenziale φ negativo. Trascureremo poi gli stati eccitati in base alle seguenti considerazioni (*).

Siano n e n' i numeri per cm^3 di nuclei di due specie nucleari (ZA) e $(Z'A')$ nello stato fondamentale, e formiamo il rapporto secondo la (9):

$$\frac{n}{n'}=\left(\frac{m}{m'}\right)^{\frac{3}{2}}\exp\left[-\alpha^*(Z-Z')+\gamma^*(A-A')-\frac{c^2}{kT}(m-m')\right]. \tag{10}$$

Se, anzichè gli stati fondamentali, si considerano stati eccitati aventi gli eccessi di energia ΔE e $\Delta E'$ sopra il livello fondamentale, nella (10) a secondo membro avremo anche un fattore f_i/f_i', ed al posto di $\exp[-(c^2/kT)(m-m')]$ comparirà il fattore:

$$\exp\left[-\frac{c^2}{kT}\left\{\left(m'-\frac{\Delta E}{c^2}\right)-\left(m'-\frac{\Delta E'}{c^2}\right)\right\}\right]. \tag{11}$$

Ora $f_i/f_i'\sim 1$ e ΔE e $\Delta E'$, nel nostro caso, sono dell'ordine di $kT\sim 10^6$ eV$\sim$ $\sim 10^{-6}$ erg, perciò $\Delta E/c^2$ e $\Delta E'/c^2$ sono dell'ordine di 10^{-27} g, mentre m ed m' valgono circa $A\cdot 1{,}7\cdot 10^{-24}$ g. Nella (11) la differenza $\Delta E/c^2-\Delta E'/c^2$ non è sempre trascurabile di fronte a $m-m'$, però gli stati eccitati più probabili sono quelli aventi $\Delta E\sim\Delta E'\sim kT$, e per questi, i termini in ΔE e in $\Delta E'$ si elidono.

Perciò ammetteremo che nel rapporto $n_{ZA}/n_{Z'A'}$ eseguito secondo la (9) il contributo dato da $\Gamma_{ZA}/\Gamma_{Z'A'}$ sia dell'ordine dell'unità. Anzi nel seguito ci occuperemo solo dei rapporti fra concentrazioni di specie nucleari diverse, perciò trascureremo il contributo degli stati eccitati ponendo $\Gamma_{ZA}=1$. Con queste approssimazioni e con la convenzione suaccennata per la temperatura, la (9) diventa:

$$n_{ZA}=2\left[2\pi m_{ZA}\frac{kT}{h^2}\right]^{\frac{3}{2}}\cdot\exp[-\alpha^*Z+\gamma^*A-\beta^*M_{ZA}], \tag{12}$$

dove m_{ZA} e M_{ZA} indicano la massa del nucleo rispettivamente in grammi e in uma, e

$$\beta^*=\beta c^2\mu=\frac{c^2\mu}{kT}, \tag{13}$$

dove μ è un fattore di conversione fra grammi ed unità di massa atomiche $(m=\mu M)$. Per il rapporto fra le concentrazioni n e n' di due specie nucleari

(*) Comunicazione verbale del prof. G. Wataghin.

(ZA) e $(Z'A')$ di masse M e M' si ha:

$$\lg \frac{n}{n'} = \frac{3}{2} \lg \frac{M}{M'} - a(Z - Z') + D(A - A') - B(M - M'), \tag{14}$$

con

$$a = \alpha^* \lg e\,, \qquad D = \gamma^* \lg e \tag{15}$$

e

$$B = \beta^* \lg e = \frac{c^2 \mu}{kT} \lg e\,. \tag{16}$$

La (14) è la formula usata nei nostri calcoli. Passando ai valori numerici, per la temperatura, otteniamo dalla (16):

$$T = \frac{1}{B} \frac{c^2 \mu \lg e}{K} = \frac{1}{B} \cdot 4{,}699 \cdot 10^{12}\ ^\circ\mathrm{K} \tag{17}$$

e per la densità, in numeri per cm^3:

$$n_{ZA} = 10^{39} \cdot 3{,}8197 \left(\frac{M}{B}\right)^{\frac{3}{2}} \cdot \exp\left[-\alpha^* Z + \gamma^* A - \beta M\right], \tag{18}$$

dove la massa M è misurata in uma, e le costanti α^*, γ^* e β^* si ricavano dalle (15) e (16) una volta fissati a, D e B.

3. – Abbiamo applicato i risultati della teoria alla ricerca della frequenza con cui si presentano gli elementi chimici nell'universo, ammettendo come stato iniziale uno stato ad alta densità ed alta temperatura seguito da una rapida espansione. Si ammette inoltre che gli elementi si siano formati in condizioni abbastanza vicine a quelle di equilibrio termodinamico, in modo da giustificare l'applicabilità delle formule usate. Durante la fase di espansione con diminuzione della temperatura e della densità, la distribuzione degli elementi resta ovviamente « congelata », dopo che la temperatura si è abbassata fino a valori per i quali le reazioni nucleari non hanno più luogo con velocità sensibile.

Abbiamo eseguito calcoli con diversi valori delle costanti a, D e B. Qui riportiamo i risultati ottenuti nel modo seguente: si assegnava a B un certo valore, $B = 5$, 10, 44, 100 e 120. Scelto B, le costanti a e D venivano determinate in un modo da adattare approssimativamente le curve ai dati sperimentali, scegliando come punti di riferimento le frequenze naturali dei nuclei ${}^{4}_{2}\mathrm{He}$, ${}^{115}_{49}\mathrm{In}$ e ${}^{235}_{92}\mathrm{U}$. In tutti i casi in cui le curve ottenute si adattavano abbastanza bene alla curva sperimentale, la densità più alta risultò quella dei

neutroni $Z = 0$, $A = 1$ e $M = 1{,}008982$. Nelle curve delle fig. 2, 3, 4, 5 la densità dei neutroni è stata riportata sull'ascissa dell'idrogeno, perchè i neutroni si sono trasformati successivamente in protoni e le curve teoriche si riferiscono alla distribuzione esistente $5 \cdot 10^9$ anni fa, mentre la curva sperimentale dà i dati attuali.

Tabella I.

a	D	B	Temperatura gradi °K	Densità neutroni degeneri g cm^{-3}
1,4287	5,4974	5	$0{,}94 \cdot 10^{12}$	$9{,}28 \cdot 10^{14}$
1,5718	10,554	10	$4{,}70 \cdot 10^{11}$	$3{,}41 \cdot 10^{14}$
2,5448	44,9385	44	$1{,}07 \cdot 10^{11}$	$4{,}68 \cdot 10^{13}$
4.303	101,6302	100	$4{,}7 \cdot 10^{10}$	$2{,}13 \cdot 10^{13}$
4.743	121,807	120	$3{,}92 \cdot 10^{10}$	$1{,}62 \cdot 10^{13}$

Nella tab. I riportiamo la temperatura e la densità dei neutroni in g·cm^{-3} corrispondenti ai valori delle costanti a, D e B. La temperatura è data dalla (17), mentre la densità dei neutroni non è stata calcolata colla (18) ma con un'altra formula, di cui diremo più avanti, in cui si tien conto della degenerazione. La fig. 1 rappresenta il logaritmo decimale della frequenza naturale degli atomi degli elementi chimici in funzione del numero atomico Z. Si è posto arbitrariamente $\log n = 10$ per il silicio. Perciò la curva dà, in scala logaritmica, il numero di atomi dell'elemento Z esistenti in natura per ogni 10^{10} atomi di silicio (per questi dati abbiamo consultato: Landolt-Börnstein: III Bd., 55, (Berlin, 1952); H. Brown: *Rev. Mod. Phys.*, **21**, 628 (1949); H. C. Urey: *Phys. Rev.*, **88**, 252 (1952)). Vi sono discordanze fra i dati sperimentali dei vari autori che si sono occupati dell'argomento. I tratti verticali nella figura rappresentano le discordanze fra i dati di Brown e quelli di

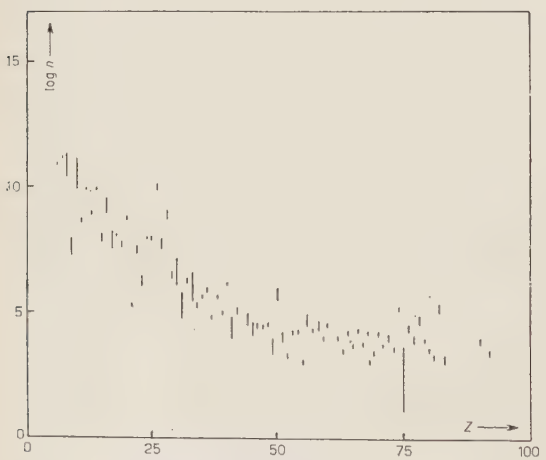

Fig. 1. – Frequenza sperimentale degli elementi chimici in funzione del numero atomico Z. Sulle ordinate è riportato il logaritmo decimale del numero di atomi di un dato elemento esistenti in natura per ogni 10^{10} atomi di silicio.

GOLDSCHMIDT. I dati sperimentali possono essere ritenuti precisi solo entro un fattore quattro, mentre un fattore due rappresenta l'errore più comune. Le fig. 2, 3, 4, 5 rappresentano le curve teoriche ottenute secondo la (14) con $B = 10$, 44, 100, 120 rispettivamente. Non abbiamo riportato la curva relativa a $B = 5$ perchè risultò praticamente uguale a quella con $B = 10$, il che significa che per temperature diverse $\gtrsim 10^{11}$ °K la distribuzione di equilibrio rimane quasi la stessa.

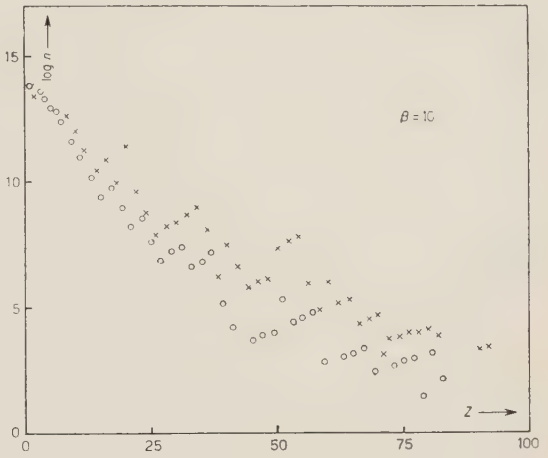

Fig. 2. – Frequenza teorica. Le crocette si riferiscono ai casi in cui l'isotopo teoricamente più abbondante ha numero di massa A pari, i cerchietti ai casi con A dispari. Analogamente per le curve delle fig. 3, 4 e 5.

Abbiamo eseguito i calcoli per tutti gli isotopi naturali, riportando sui grafici, per un dato valore di Z, la frequenza di quell'isotopo che risultò il più abbondante teoricamente. Per tutti i valori di B abbiamo sempre trovato che l'isotopo più abbondante era quello di massa maggiore. Dall'osservazione delle curve e della tabella risulta subito che la distribuzione teorica si approssima tanto meglio a quella sperimentale quanto più la temperatura è elevata. Ma le densità della materia corrispondenti alle distribuzioni accettabili sono vicine alle densità dei nuclei stessi (circa $10^{14}\,g \cdot cm^{-3}$) e perciò sembrano troppo alte. Viceversa, a temperature più basse corrispondono densità accettabili ma la distribuzione dei nuclei risulta troppo discordante da quella sperimentale.

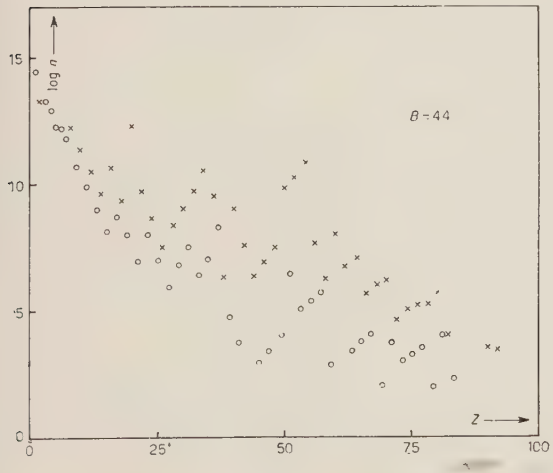

Fig. 3.

Considerazioni particolari.

a) Nei nostri calcoli risultò sempre che l'isotopo più abbondante di un dato elemento era quello avente il maggior numero di neutroni. Questo è gene-

ralmente in contraddizione con i dati sperimentali, specialmente per gli elementi leggeri. Osserviamo però che le percentuali sperimentali dei vari isotopi di un dato elemento si riferiscono a dati terrestri, mentre i dati stellari di origine spettroscopica non permettono di conoscere in generale le percentuali relative esistenti nelle stelle. Nelle fig. 2, 3, 4, 5, le crocette si riferiscono ai casi in cui l'isotopo teoricamente più abbondante aveva numero di massa A pari, i cerchietti ai casi con A dispari. Nella fig. 2 si vede che in genere un nucleo con A pari è più abbondante dei due nuclei adiacenti aventi A dispari, e ciò corrisponde ai dati sperimentali (vedere, per esempio, il LANDOLT-BÖRNSTEIN, op. cit.).

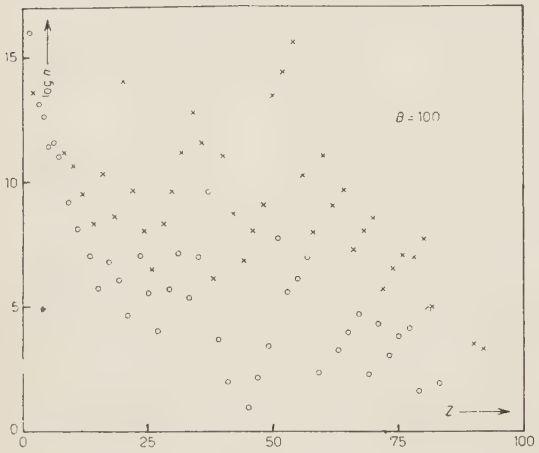

Fig. 4.

b) Abbiamo calcolato solo coi nuclei stabili e con gli instabili a periodo maggiore $> 3 \cdot 10^9$ anni. È chiaro che in un calcolo rigoroso si dovrebbe tener conto anche dei nuclei instabili a breve periodo, perchè essi hanno vite medie ben maggiori della durata della fase iniziale esplosiva ad altissima temperatura ammessa nell'ipotesi dell'origine degli elementi. Il tenerne conto avrebbe richiesto un notevole allungamento dei calcoli. Tuttavia riteniamo che calcolando anche coi nuclei instabili situati sopra e sotto la linea di fondo valle nelle tavole (ZA), avremmo ottenuto soltanto un aggiustamento nei particolari delle curve, ma non un cambiamento dell'andamento generale. Basiamo la nostra convinzione sul fatto che nell'eseguire i calcoli abbiamo sempre trovato delle variazioni uniformi delle abbondanze nel

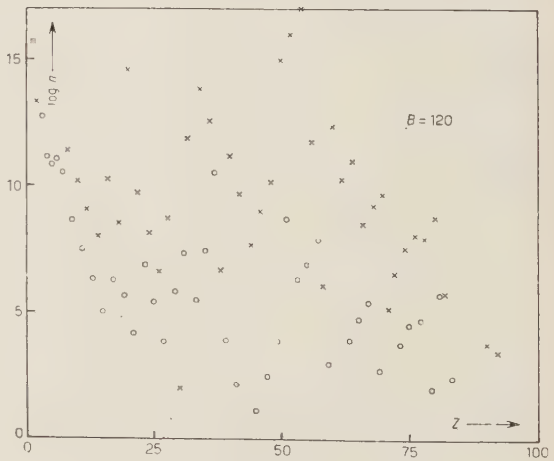

Fig. 5.

passare da un nucleo (Z, A) ad un nucleo $(Z, A+1)$ o ad un altro nucleo adiacente, di modo che i logaritmi dei rapporti

$$n_{Z,A+1}/n_{Z,A} \quad \text{e} \quad n_{Z,A}/n_{Z+1,A}$$

mantengono un valore quasi costante per tutti i valori di Z e di A. Perciò considerando per tutti gli elementi i nuclei con un neutrone in più (o due) oltre quello stabile avente A massimo, avremmo trovato che le abbondanze variavano quasi nel medesimo rapporto per ogni Z. Per esempio, con $B = 10$, il $\log [n_{Z,A+1}/n_{Z,A}]$ oscilla generalmente fra 0,5 e 1 per ogni Z, cosicchè la considerazione dei nuclei instabili e del loro modo di disintegrarsi porterebbe solo ad un aggiustamento nei particolari.

c) La (14) si riferisce non alla distribuzione attuale dei nuclei ma a quella originaria, cioè a quella di circa $5 \cdot 10^9$ anni addietro. Nel calcolare le costanti a e D in base alle frequenze naturali dell'He, In e U abbiamo tenuto conto della variazione della quantità di U per disintegrazione e della trasformazione di ${}_1^1\mathrm{H}$ in ${}_2^4\mathrm{He}$, ma tali variazioni si sono rivelate insignificanti di fronte alle incertezze sperimentali delle abbondanze (per esempio, il rapporto idrogeno/elio varia da 5 a 30 a seconda degli autori (*)). La considerazione di tali variazioni non altera assolutamente l'andamento generale delle curve teoriche.

d) Abbiamo detto che a temperature diverse $\gtrsim 10^{11}$ °K la distribuzione teorica rimane quasi invariata. La ragione è la seguente. Nella (14) il termine $-a(Z-Z') + D(A-A') - B(M-M')$ può scriversi:

$$-a(Z-Z') + D(D-B)(A-A') - B[(M-A)-(M'-A')] .$$

Ora, poichè a e $D-B$ risultano dell'ordine dell'unità, il termine $-B[(M-A)-(M'-A')]$ per valori di $B < 10$ viene ad avere importanza trascurabile rispetto agli altri due. Infatti la curva con $B = 5$ risultò praticamente uguale a quella con $B = 10$.

e) La (12) è stata ottenuta trascurando gli stati eccitati e ponendo nella (7) $E = mc^2 + p^2/2m$ al posto di E. Mediante la (6) l'integrale della (7) diventa:

$$n_{ZA} = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 \, dp}{\exp[y] \exp[p^2/2mkT] \pm 1} , \tag{19}$$

(*) Vedi, per esempio, i dattiloscritti: *Conference on the abundance of the elements*, Yerkes Observatory, William Bay, Wisconsin, November 6, 7, 8 (1952).

dove

$$y = \alpha^* Z - \gamma^* A + \beta^* M = \frac{1}{\lg e}[aZ - DA + BM] .$$

Per i neutroni si ha degenerazione perchè il fattore e^v nell'integrando della (19), per $B = 5$, 10, 44, 100, 120 assume rispettivamente i valori 0,353; 0,344; 0,286; 0,185; 0,186. Perciò la (12) ottenuta trascurando ± 1 a denominatore della (19) non è più valida.

Per i neutroni, integrando la (19) con l'approssimazione:

$$\int_0^\infty \frac{u^{\frac{1}{2}}\,du}{(1/x)\exp[u/kT] + 1} \approx \frac{2}{3}(kT\ \ln x)^{\frac{3}{2}}$$

si ottiene:

$$n = f \cdot \frac{8\pi}{3}\left(\frac{2\pi kT}{h^2}\right)^{\frac{3}{2}} \cdot (\gamma^* - \beta^* M)^{\frac{3}{2}} ,$$

dove è stato introdotto il fattore $f = 2$, peso statistico del neutrone. Con le posizioni già fatte, si trova:

$$n = f \cdot 10^{39} \cdot 2.8734 \left(\frac{M}{B}\right)^{\frac{3}{2}} \cdot [\gamma^* - \beta^* M]^{\frac{3}{2}} . \tag{20}$$

Le densità riportate nella tab. I sono state calcolate con questa formula e risultarono circa la terza parte di quelle calcolate con la (18).

f) Assumiamo il potenziale gravitazionale φ normalizzato all'infinito nel solito modo, e ammettiamo che φ sia approssimativamente costante nel volume occupato dall'insieme delle particelle nello stato iniziale. Il primo membro della (3) non può essere negativo, perciò dobbiamo avere $1 + \varphi/c^2 \geqslant 0$, cioè, essendo $\varphi < 0$,

$$|\varphi| \leqslant c^2 . \tag{21}$$

Per una massa sferica m, di raggio R e densità media ϱ, si ottiene, come ordine di grandezza,

$$|\varphi| \simeq G\frac{m}{R} = G\frac{4}{3}\pi R^2 \varrho , \tag{22}$$

dove G è la costante gravitazionale. Si ammetta una densità $\varrho \sim 10^{13}$ g·cm^{-3}, in tal caso dalle (21) e (22) risulta $R \lesssim 10^7$ cm, e quindi per la massa della

sfera otteniamo, come ordine di grandezza

$$m \lesssim 2{,}7 \cdot 10^{35}\ \mathrm{g}\,. \tag{23}$$

Questo limite superiore per m è in accordo coi dati astrofisici sulle masse delle stelle.

In conclusione: l'ipotesi della formazione degli elementi chimici in condizioni di equilibrio termodinamico, trattata con le usuali teorie della termodinamica statistica, conduce a risultati teorici in disaccordo coi dati sperimentali. Sembra pertanto che, pur non rinunciando al metodo termodinamico statistico, sia necessario apportare idee nuove, ed in particolare tener conto anche delle irreversibilità essenziali nella dinamica dei processi così rapidi come quelli sopra considerati.

Ringrazio il prof. G. Wataghin per avermi consigliato l'argomento e per i suoi consigli ed incoraggiamenti.

In un lavoro successivo darò un resoconto dettagliato sui valori delle masse dei nuclei atomici usate in questi calcoli.

SUMMARY (*)

The formulae of statistical thermodynamics, obtained taking count of the relativistic energy of the particles and of the creation of pairs, are applied to the investigation of the distribution in the universe of the chemical elements, assuming that the elements have been formed in conditions sufficiently near those of thermodynamical equilibrium, so as to justify the assumption of the approximate validity of these formulae. Starting from the attempts to deduce from a thermodynamical theory the observed distribution of elements, one finds that an accord with the observational data is obtainable only for densities $\varrho \gtrsim 10^{13}$ g/cm^3; moreover one obtains an upper limit for the total mass $m \lesssim 10^{35}$ g, in accord with the astrophysical data concerning the stellar masses.

(*) *Traduzione a cura della Redazione.*

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The Effect of Nuclear Compressibility on the Nuclear Photoeffect.

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(ricevuto il 10 Settembre 1954)

Summary. — A general two-fluid nuclear model is formulated for the case in which the nuclear energy density depends only on the local values of nucleon densities and not on their derivatives. Hydrodynamical equations, together with a generalised Laplace's equation, are derived taking account of compressibility. The energies corresponding to the lowest dipole oscillation modes are calculated for a few nuclei and compared with Jensen-Steinwedel's and Danos' results for the resonance energies for photon absorption. Better agreement with the experimental data is obtained, especially, as might be expected, for moderately heavy nuclei.

1. – The purpose of the present calculations is to investigate the importance of the compressibility of nuclear matter for the nuclear photoeffect. We have chosen this particular phenomenon because, as it is a relatively high energy process, it would be more influenced by the compressibility than ground state or low-lying excited state properties.

It was usually assumed that, at low energies (a few MeV), photon absorption by nuclei was mainly due to dipole and quadrupole transitions, the importance of the former being greatly reduced by correlation between the motions of the nucleons. But, in 1948, it became clear that the experimental values of the cross-sections were much too high and, besides, it appeared that the cross-section curves exhibited resonance maxima. These maxima could not be explained as due to competition between different photonuclear reactions as had been suggested and seemed to indicate that, at higher energies (~ 20 MeV), correlations between the motion of the nucleons were not so important. Dipole

(*) On leave of absence from Universidade do Porto, Portugal.

absorption could then play a dominant role and GOLDHABER and TELLER [1] proposed a mechanism for photon absorption (*), in which there was anti-correlation between neutrons and protons. They used a model in which a rigid spherical domain of incompressible proton fluid could be displaced relatively to a similar neutron domain. Although some of the constants appearing in their treatment were rather uncertain, the model gave values of the right order of magnitude for the cross-sections and resonance energies for photon absorption. These came out to exhibit an $A^{-\frac{1}{6}}$ mass number dependence.

JENSEN and STEINWEDEL [3], following a suggestion of GOLDHABER and TELLER [1] developed a two-fluid model, assuming incompressibility for the nuclear matter, a sharp, rigid, spherical nuclear surface, and uniform static proton and neutron distributions. Their treatment had the advantage of not introducing such uncertain constants (as ε of GOLDHABER and TELLER's paper) and gave an $A^{-\frac{1}{3}}$ dependence for the resonance energy for photon absorption which, at that time, was believed to be in better agreement with the experimental results. However, their results were about 30% too small when compared with the experimental values, and the resonance widths were interpreted as due to a viscosity numerically chosen so as to fit the average experimental widths. DANOS [4] tried to refine Jensen-Steinwedel's model by considering the influence of Coulomb forces and non-homogeneity of the static proton distribution, but without giving up the assumption of incompressibility. Only the second effect seemed to be significant, and though his results are nearer the experimental values, they are still about 20% too small for heavy nuclei.

FLÜGGE and WOESTE investigated ground state and excited state properties of nuclei using the liquid drop model; but, while for the ground state (FLÜGGE and WOESTE [5]) compressibility and the existence ot two fluids were simultaneously taken into account, for the excited states WOESTE [6] considered only either one or the other separately. His treatment of the two-fluid liquid drop allows for surface deformations (neglected by DANOS) but again does not take compressibility into account. From his calculations it came out that a compressible one-fluid liquid drop has dipole oscillation modes corresponding to about the same energies as those of an incompressible two-fluid liquid drop. This has also been shown by MARX [7] who gave a treatment of the oscillation modes of a compressible one-fluid liquid drop, using some simplifying assumptions not introduced by WOESTE.

LEVINGER and BETHE [8] investigated dipole transitions in the nuclear photoeffect without using any definite model for the absorption mechanism. Starting from independent particle wave functions and assuming two-body central forces between the nucleons they could derive, by sum rules only, the

(*) N. BOHR [2] had suggested such a mechanism responsible for the separation of the nuclear centers of mass and charge.

integrated cross-section and mean photon energy for absorption by dipole processes. The results did not depend sensitively on the shape of the neutron-proton potential and contained a parameter x, the fraction of atractive exchange force for the neutron-proton potential, the value of which should be chosen by comparison with the experimental results. But the resonance character of photon absorption did not come out of their calculations.

Reifman [9] has introduced a model combining, like A. Bohr's, collective and individual-particle aspects of nuclear structure. Although using some oversimplifying assumptions (infinitely deep well potential, nucleons interacting only through their coupling to surface deformations), he was able to account not only for the resonance character of photon absorption but also for the order of magnitude of the experimental resonance widths. He seems to have taken for granted that the collective oscillations of a rarefied Fermi gas can be represented by the Hamiltonian of an incompressible liquid drop —which is not at all obvious; still, an approach along his lines (using a well potential of finite depth) could probably give an explanation for the non-isotropic angular distribution of the products of some photonuclear reactions, observed at rather high energies (~ 100 MeV) (*).

In the following we develop a two-fluid hydrodynamical model, allowing for compressibility, and compare our results with the experimental data and the results of Jensen, Steinwedel and Danos.

2. – We will consider the case in which the nuclear energy density depends only on the local values of proton and neutron densities. Although, as already remarked by Swiatecki [13], the nuclear energy density must also depend on derivatives of nucleon densities, such dependence leads to hydrodynamical equations which it does not seem possible to integrate. If we assume further nearly homogeneous proton and neutron distributions, such that we can represent the nuclear energy density $\varepsilon(\varrho_n, \varrho_p)$ by its series development around the values of ϱ_n and ϱ_p for a standard heavy nucleus, up to second order terms, we can write, following Swiatecki,

$$\varepsilon(\varrho_n, \varrho_p) = \varepsilon_s + P\varphi + P'\psi + \tfrac{1}{2}Q\varphi^2 + \tfrac{1}{2}S\psi^2 + R\varphi\psi ,$$

(*) In a letter to the *Physical Review*, Ferentz, Gell-Mann and Pines [10] report on the results of a calculation based on Bohm and Pines' collective description of fermion interactions. They get for the resonance energy for photon absorption $E = W_0 A^{-\frac{1}{3}}$ with $W_0 \simeq 80$ MeV; this numerical value is said to be only a rough estimate and the values used for several constants are not quoted. However, so far, attempts to excite plasma oscillations of the type predicted by Bohm and Gross [11] have been unsuccessful, even with experiments specially designed for this purpose (Looney and Brown [12]).

with

$$\varrho_n = Mn_n, \quad \varrho_p = Mn_p, \quad \varepsilon_s = \varepsilon(n_0, n_0), \quad \varphi = \tfrac{1}{2}(n_n + n_p) - n_0, \quad \psi = \tfrac{1}{2}(n_n - n_p)$$

and $n_0 = 3/8\pi r_0^3$, the latter being the proton and neutron densities for a standard heavy nucleus.

The terms in $\varphi\psi$ and ψ must be omitted because, roughly speaking, they would give, after integration over the nuclear volume, terms proportional to $N^2 - Z^2$ and $N - Z$ respectively, which do not occur in the semi-empirical mass formula. In fact, this can be checked by a variational calculation similar to SWIATECKI's [13].

Therefore we will use for the nuclear energy density

$$\varepsilon(\varrho_n, \varrho_p) = \varepsilon_s + P\varphi + \tfrac{1}{2}Q\varphi^2 + \tfrac{1}{2}S\psi^2, \tag{1}$$

where $\frac{1}{2}Q\varphi^2$ and $\frac{1}{2}S\psi^2$ represent the compressibility and neutron-proton asymmetry contributions; it will turn out that P will not occur in our final formulae.

This expression leads to static proton and neutron distributions which are the same as those given by FLÜGGE and WOESTE [5] and resemble the distributions recently suggested by JOHNSON and TELLER [14]. A more refined model, in which the dependence of ε on derivatives of nucleon densities were included, would possibly not require the assumption of a sharp nuclear surface, and might give proton and neutron distributions closer to those proposed by JOHNSON and TELLER.

We will derive the hydrodynamical equations for our problem from Hamilton's principle

$$\int_{t_0}^{t_1} \mathrm{d}t[\delta(T - U) + \delta W] = 0, \tag{2}$$

with

$$T = \int_{V(t)} (\tfrac{1}{2}\varrho_p \boldsymbol{v}_p^2 + \tfrac{1}{2}\varrho_n \boldsymbol{v}_n^2)\,\mathrm{d}\tau, \tag{3}$$

$$U = \int_{V(t)} \varepsilon(\varrho_n, \varrho_p)\,\mathrm{d}\tau + \int_{S(t)} \gamma\,\mathrm{d}\sigma, \tag{4}$$

$$\delta W = \int_{V(t)} \frac{e}{M}\varrho_p(\boldsymbol{E} + \boldsymbol{v}_p \wedge \boldsymbol{B})\cdot\delta\boldsymbol{s}_p\,\mathrm{d}\tau, \tag{5}$$

and $\delta\boldsymbol{s}_p = \delta\boldsymbol{s}_n = 0$ for $t = t_0$ and $t = t_1$. Although in our case $\boldsymbol{E}$, $\boldsymbol{B}$ represent the electromagnetic field due to the proton distribution, equation (5) is quite

general — $\boldsymbol{E}$, $\boldsymbol{B}$ could be any electromagnetic field. Our approximation will be to consider only the effect of electrostatic forces.

The continuity equation will be taken into account by the relations

$$\delta\varrho_p = -\operatorname{div}(\varrho_p \delta \boldsymbol{s}_p)\,; \qquad \delta\varrho_n = -\operatorname{div}(\varrho_n\,\delta \boldsymbol{s}_n) \tag{6}$$

and, since the virtual displacements must not violate the assumption we shall make that protons and neutrons fill the same domain, we must require that

$$\delta \boldsymbol{s}_p \cdot \boldsymbol{\nu} = \delta \boldsymbol{s}_n \cdot \boldsymbol{\nu} \tag{7}$$

on the nuclear surface S, for any t. ($\boldsymbol{\nu}$ is the unit vector normal to the surface). As we shall need $\delta \boldsymbol{v}(\boldsymbol{r}, t)$ we begin by calculating it.

Considering two operators D and Δ

$$D = d + \boldsymbol{v}\,\mathrm{d}t\cdot\operatorname{grad}\,; \qquad D\boldsymbol{r} = \boldsymbol{v}(\boldsymbol{r}, t)\,\mathrm{d}t\,,$$

$$\Delta = \delta + \delta\boldsymbol{s}\cdot\operatorname{grad}\,; \qquad \Delta\boldsymbol{r} = \delta\boldsymbol{s}(\boldsymbol{r}, t)\,,$$

we can write, since they commute, $\Delta D\boldsymbol{r} = D\Delta\boldsymbol{r}$.

From this follows

$$\delta\boldsymbol{v}(\boldsymbol{r}, t) = \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\operatorname{grad}\right)\delta\boldsymbol{s} - (\delta\boldsymbol{s}\cdot\operatorname{grad})\,\boldsymbol{v}\,. \tag{8}$$

As

$$\delta\int_V F(\boldsymbol{r})\,\mathrm{d}\tau = \int_V \delta F(\boldsymbol{r})\,\mathrm{d}\tau + \int_S F(\boldsymbol{r})\delta\boldsymbol{s}\cdot\boldsymbol{\nu}\,\mathrm{d}\sigma\,, \tag{9}$$

we can write

$$\delta T_p = \int_V (\tfrac{1}{2}\delta\varrho_p \boldsymbol{v}_p^2 + \varrho_p \boldsymbol{v}_p\cdot\delta\boldsymbol{v}_p)\,\mathrm{d}\tau + \int_S \tfrac{1}{2}\varrho_p \boldsymbol{v}_p^2\,\delta\boldsymbol{s}\cdot\boldsymbol{\nu}\,\mathrm{d}\sigma\,.$$

Substituting $\delta\varrho_p$ and $\delta\boldsymbol{v}_p$ from (6) and (8) we have, after some straightforward transformations,

$$\delta T_p = -\int_V \varrho_p\left[\frac{\partial\boldsymbol{v}_p}{\partial t} + (\boldsymbol{v}_p\cdot\operatorname{grad})\,\boldsymbol{v}_p\right]\cdot\delta\boldsymbol{s}_p\,\mathrm{d}\tau \tag{10}$$

and similarly for δT_n.

For δU we can write, taking into account (7),

$$\delta U = \int_V \delta\varepsilon(\varrho_n, \varrho_p)\mathrm{d}\tau + \int_S \varepsilon(\varrho_n, \varrho_p)\delta\boldsymbol{s}\cdot\boldsymbol{\nu}\,\mathrm{d}\sigma + \gamma\delta S$$

or, because of

$$\delta S = \int_S C_m \delta \boldsymbol{s} \cdot \boldsymbol{\nu} \, \mathrm{d}\sigma ,$$

where C_m is the mean curvature of the nuclear surface,

$$\delta U = \int_V \left(\frac{\partial \varepsilon}{\partial \varrho_n} \delta \varrho_n + \frac{\partial \varepsilon}{\partial \varrho_p} \delta \varrho_p \right) \mathrm{d}\tau + \int_S (\varepsilon + \gamma C_m) \delta \boldsymbol{s} \cdot \boldsymbol{\nu} \, \mathrm{d}\sigma ,$$

or finally

$$\delta U = \int_V \left[\varrho_n \operatorname{grad} \left(\frac{\partial \varepsilon}{\partial \varrho_n} \right) \cdot \delta \boldsymbol{s}_n + \varrho_p \operatorname{grad} \left(\frac{\partial \varepsilon}{\partial \varrho_p} \right) \cdot \delta \boldsymbol{s}_p \right] \mathrm{d}\tau + \int_S \left[\varepsilon - \varrho_n \frac{\partial \varepsilon}{\partial \varrho_n} \quad \varrho_p \frac{\partial \varepsilon}{\partial \varrho_p} + \gamma C_m \right] \delta \boldsymbol{s} \cdot \boldsymbol{\nu} \, \mathrm{d}\sigma . \tag{11}$$

Substituting (10), (11) and (5) in (2), we get generalised Euler equations

$$\frac{\partial \boldsymbol{v}_p}{\partial t} + (\boldsymbol{v}_p \cdot \operatorname{grad}) \boldsymbol{v}_p = - \operatorname{grad} \frac{\partial \varepsilon}{\partial \varrho_p} + \frac{e}{M} (\boldsymbol{E} + \boldsymbol{v}_p \wedge \boldsymbol{B}) , \tag{12}$$

$$\frac{\partial \boldsymbol{v}_n}{\partial t} + (\boldsymbol{v}_n \cdot \operatorname{grad}) \boldsymbol{v}_n = - \operatorname{grad} \frac{\partial \varepsilon}{\partial \varrho_n} , \tag{13}$$

together with an equation, valid on S, which is a generalisation of Laplace's relation between pressure, surface tension and mean curvature:

$$\varepsilon - \varrho_n \frac{\partial \varepsilon}{\partial \varrho_n} - \varrho_p \frac{\partial \varepsilon}{\partial \varrho_p} + \gamma C_m = 0 . \tag{14}$$

For the simple case of one compressible fluid, with $\varepsilon = - \varrho \int p(\varrho) \, \mathrm{d}(1/\varrho)$, our equations go over into Euler and Laplace's equations.

Naturally, besides (12) and (13), we have the continuity equations

$$\operatorname{div} (\varrho_p \boldsymbol{v}_p) + \frac{\partial \varrho_p}{\partial t} = 0 ; \qquad \operatorname{div} (\varrho_n \boldsymbol{v}_n) + \frac{\partial \varrho_n}{\partial t} = 0 \tag{15}$$

as well as, the equations

$$\begin{cases} \operatorname{div} \boldsymbol{E} = \dfrac{1}{\varepsilon_0} \cdot \dfrac{e}{M} \varrho_p ; & \dfrac{1}{\mu_0} \operatorname{curl} \boldsymbol{B} = \dfrac{e}{M} \varrho_p \boldsymbol{v}_p + \varepsilon_0 \dfrac{\partial \boldsymbol{E}}{\partial t} , \\ \boldsymbol{E} = - \operatorname{grad} V - \dfrac{\partial \boldsymbol{A}}{\partial t} ; & \Delta V - \varepsilon_0 \mu_0 \dfrac{\partial^2 V}{\partial t^2} = - \dfrac{1}{\varepsilon_0} \dfrac{e}{M} \varrho_p . \end{cases} \tag{16}$$

We will now assume irrotational proton and neutron motions (for comments on this assumption see BOHR and MOTTELSON [15]) and denote by Φ_p and Φ_n the corresponding velocity potentials. Substituting ε from (1), equations (12), (13) and (15) now become

$$\text{(17)}\qquad \operatorname{grad}\frac{\partial\Phi_p}{\partial t}+\tfrac{1}{2}\operatorname{grad}(\operatorname{grad}\Phi_p)^2+(Q-S)/4M^2\operatorname{grad}\varrho_n+$$
$$+(Q+S)/4M^2\operatorname{grad}\varrho_p=\frac{e}{M}(\boldsymbol{E}+\operatorname{grad}\Phi_p\wedge\boldsymbol{B})\,,$$

$$\text{(18)}\qquad \operatorname{grad}\frac{\partial\Phi_n}{\partial t}+\tfrac{1}{2}\operatorname{grad}(\operatorname{grad}\Phi_n)^2+(Q+S)/4M^2\operatorname{grad}\varrho_n+$$
$$+(Q-S)/4M^2\operatorname{grad}\varrho_p=0\,,$$

$$\text{(19)}\qquad \frac{\partial\varrho_p}{\partial t}+\varrho_p\Delta\Phi_p+\operatorname{grad}\varrho_p\cdot\operatorname{grad}\Phi_p=0\,,$$

$$\text{(20)}\qquad \frac{\partial\varrho_n}{\partial t}+\varrho_n\Delta\Phi_n+\operatorname{grad}\varrho_n\cdot\operatorname{grad}\Phi_n=0\,.$$

From equations (17), (19) and (18), (20), taking into account (16), one can derive

$$\text{(21)}\qquad \tfrac{1}{2}\Delta(\operatorname{grad}\Phi_p)^2-\frac{\partial}{\partial t}\left(\frac{1}{\varrho_p}\operatorname{grad}\varrho_p\cdot\operatorname{grad}\Phi_p\right)+\frac{1}{\varrho_p^2}\left(\frac{\partial\varrho_p}{\partial t}\right)^2+\frac{e^2}{M^2}\mu_0\varrho_p(\operatorname{grad}\Phi_p)^2+$$
$$+\frac{e}{M}\varepsilon_0\mu_0\frac{\partial\boldsymbol{E}}{\partial t}\cdot\operatorname{grad}\Phi_p-\frac{1}{\varrho_p}\cdot\frac{\partial^2\varrho_p}{\partial t^2}-\frac{e^2}{M^2}\cdot\frac{1}{\varepsilon_0}\varrho_p+(Q-S)/4M^2\Delta\varrho_n+$$
$$+(Q+S)/4M^2\Delta\varrho_p=0\,,$$

$$\text{(22)}\qquad \tfrac{1}{2}\Delta(\operatorname{grad}\Phi_n)^2-\frac{\partial}{\partial t}\left(\frac{1}{\varrho_n}\operatorname{grad}\varrho_n\cdot\operatorname{grad}\Phi_n\right)+\frac{1}{\varrho_n^2}\left(\frac{\partial\varrho_n}{\partial t}\right)^2-$$
$$-\frac{1}{\varrho_n}\cdot\frac{\partial^2\varrho_n}{\partial t^2}+(Q+S)/4M^2\Delta\varrho_n+(Q-S)/4M^2\Delta\varrho_p=0\,.$$

Describing the moving nuclear surface in the usual way,

$$R(\theta,\varphi,t)=R_0[1+\sum_{l,m}a_{lm}(t)P_l^m(\theta)\,e^{im\varphi}]$$

with the additional assumption $|a_{lm}|\ll 1$, it is easy to see that the 1st, 3rd, 4th and 5th terms of (21) and the 1st and 3rd of (22) are of second order in the a_{lm}.

The second terms would give first order contributions

$$-\frac{1}{\varrho_p^{\text{stat}}}\cdot\frac{\partial}{\partial r}\varrho_p^{\text{stat}}\cdot\frac{\partial}{\partial t}\left(\frac{\partial\Phi_p}{\partial r}\right)\quad\text{and}\quad-\frac{1}{\varrho_n^{\text{stat}}}\cdot\frac{\partial}{\partial r}\varrho_n^{\text{stat}}\cdot\frac{\partial}{\partial t}\left(\frac{\partial\Phi_n}{\partial r}\right)\quad\text{respectively.}$$

If the static (ground state) distributions are nearly homogeneous we will be justified in neglecting also those terms; in the same way we will be allowed to drop the last terms in equations (19) and (20) and to replace the densities in the second terms of those equations by some average values ϱ_p^0 and ϱ_n^0. However, one should probably emphasize that the equations, thus simplified, do not correspond to a consistent approximation to first order in the a_{lm}. Finally we can neglect the 2nd term in the last equation of (16) although it would be no more difficult to handle the complete equation.

We are then left with a system of partial differential equations for ϱ_p and ϱ_n

$$\text{(23)} \qquad \begin{cases} a\,\Delta\varrho_p + b\,\Delta\varrho_n + c_p \dfrac{\partial^2}{\partial t^2}\varrho_p + d\varrho_p = 0 \\ b\,\Delta\varrho_p + a\,\Delta\varrho_n + c_n \dfrac{\partial^2}{\partial t^2}\varrho_n = 0 \end{cases}$$

and with the following equations which determine the velocity potentials Φ_p, Φ_n and the electrostatic potential inside and outside the nucleus V_i, V_0:

$$\text{(24)} \qquad \Delta\Phi_p = c_p \frac{\partial\varrho_p}{\partial t}\,; \qquad \Delta\Phi_n = c_n \frac{\partial\varrho_n}{\partial t}\,,$$

$$\text{(25)} \qquad \Delta V_i = -\frac{e}{M\varepsilon_0}\cdot\varrho_p\,; \qquad \Delta V_0 = 0$$

where

$$a = (Q+S)/4M^2, \quad b = (Q-S)/4M^2, \quad c_p = -1/\varrho_p^0, \quad c_n = -1/\varrho_n^0, \quad d = -e^2/M^2\varepsilon_0.$$

In order to integrate (23) we will separate variables in $\varrho_p(r,\theta,\varphi,t)$, $\varrho_n(r,\theta,\varphi,t)$ as follows:

$$\varrho_p = \sum_{l,m,\omega_l} P_l^m(\theta)\, e^{im\varphi} e^{i\omega_l t} f_p^{lm\omega_l}(r)\,,$$

$$\varrho_n = \sum_{l,m,\omega_l} P_l^m(\theta)\, e^{im\varphi} e^{i\omega_l t} f_n^{lm\omega_l}(r)\,.$$

Substituting in (23) and dropping, for the time being, the indices m, ω_l, one gets

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+1)}{r^2}\right] f_n^l(r) + aAf_n^l(r) + bBf_p^l(r) = 0\,,$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+1)}{r^2}\right] f_p^l(r) - aBf_p^l(r) - bAf_n^l(r) = 0\,,$$

with

$$A(\omega) = \frac{c_n\omega^2}{b^2-a^2}\,; \qquad B(\omega) = \frac{d-c_p\omega^2}{b^2-a^2}\,.$$

If we make

$$B_l \equiv \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} + 1 - \frac{l(l+1)}{r^2}$$

($B_l j_l(r) = B_l n_l(r) = 0$, where $j_l(r)$ and $n_l(r)$ are the spherical Bessel and Neumann functions of order l), we can rewrite our equations as

$$B_l f_n^l(r) + (aA - 1) f_n^l(r) + bB f_p^l(r) = 0\,,$$
$$B_l f_p^l(r) - (aB + 1) f_p^l(r) - bA f_n^l(r) = 0\,.$$

It then turns out that both $f_p^l(r)$ and $f_n^l(r)$ satisfy the fourth order differential equation

$$(B_l^2 + mB_l + n)u = 0\,, \tag{26}$$

with

$$m = aA - aB - 2\,; \qquad n = b^2 AB - (aB + 1)(aA - 1)\,.$$

In order to solve (26) we look for eigenfunctions of the operator B_l. Since

$$B_l j_l(kr) = (1 - k^2) j_l(kr)\,,$$
$$B_l n_l(kr) = (1 - k^2) n_l(kr)\,,$$

B_l has a continuous eigenvalue spectrum and, in order to find particular solutions of (26), it will be enough to pick out the values of k which are solutions of the characteristic equation

$$(1 - k^2)^2 + m(1 - k^2) + n = 0\,, \tag{27}$$

that is

$$k^2 = \tfrac{1}{2}\left[m + 2 \pm \sqrt{m^2 - 4n}\right].$$

Therefore we have four values for k, namely $k_1(\omega)$; $-k_1(\omega)$; $k_2(\omega)$; $-k_2(\omega)$.

It is easy to see that the radicand is positive for any ω, so that $k_1(\omega) \neq k_2(\omega)$; besides, one of them is always real and the other imaginary for $\omega^2 < d/c_p$, real otherwise.

As

$$j_l(-x) = (-1)^l j_l(x)\,; \qquad n_l(-x) = (-1)^{l+1} n_l(x)\,,$$

we have, in fact, four independent solutions of (26):

$$j_l(k_1 r)\,, \quad j_l(k_2 r)\,, \quad n_l(k_1 r)\,, \quad n_l(k_2 r)\,.$$

The last two particular solutions have a singularity at $r = 0$, so that we must write the solution as

$$f_p^{lm\omega_l}(r) = D_{(1)p}^{lm\omega_l} j_l(k_1^{\omega_l} r) + D_{(2)p}^{lm\omega_l} j_l(k_2^{\omega_l} r) ,$$
$$f_n^{lm\omega_l}(r) = D_{(1)n}^{lm\omega_l} j_l(k_1^{\omega_l} r) + D_{(2)n}^{lm\omega_l} j_l(k_2^{\omega_l} r) .$$

The static (spherically symmetric) nucleon densities correspond obviously to $\omega = 0$. In this case

$$k_1^2 = 0 ; \quad k_2^2 = -\alpha^2 ; \quad j_0(k_1 r) = 1 ; \quad j_0(k_2 r) = (\sinh \alpha r)/\alpha r$$

with

$$\alpha^2 = e^2/\varepsilon_0(1/Q + 1/S) .$$

One can easily obtain the static distributions given by Flügge and Woeste [5]

$$\varrho_p^{\text{stat}} = C_p (\sinh \alpha r)/\alpha r ; \qquad \varrho_n^{\text{stat}} = C_n \quad b/a \varrho_p^{\text{stat}}$$

where the parameter α, which characterises the proton distribution, increases when we take account of the compressibility, i.e. when we take a finite value for Q. It is convenient to write down the nucleon densities as sums of the static values and the deviations from them. We can then write

$$(28) \quad \begin{cases} \varrho_p = C_p \dfrac{\sinh \alpha r}{\alpha r} + \sum\limits_{l,m,\omega_l} P_l^m(\theta) e^{im\varphi} e^{i\omega_l t} [D_{(1)p}^{lm\omega_l} j_l(k_1^{\omega_l} r) + D_{(2)p}^{lm\omega_l} j_l(k_2^{\omega_l} r)] \\ \varrho_n = C_n - \dfrac{b}{a} C_p \dfrac{\sinh \alpha r}{\alpha r} + \sum\limits_{l,m,\omega_l} P_l^m(\theta) e^{im\varphi} e^{i\omega_l t} [D_{(1)n}^{lm\omega_l} j_l(k_1^{\omega_l} r) + D_{(2)n}^{lm\omega_l} j_l(k_2^{\omega_l} r)] . \end{cases}$$

However, the four sets of constants $D_{(1)p}$, $D_{(2)p}$, $D_{(1)n}$, $D_{(2)n}$ are not independent. Substituting (28) in (23), besides verifying that the time-independent parts of the equations vanish identically, one gets

$$(29) \quad \begin{cases} (ak_1^2 + c_p\omega^2 - d) D_{(1)p} + bk_1^2 D_{(1)n} = 0 , \\ bk_1^2 D_{(1)p} + (ak_1^2 + c_n\omega^2) D_{(1)n} = 0 . \end{cases}$$

and similar relations between $D_{(2)p}$ and $D_{(2)n}$. But, as one should expect, the two equations in (29) are not distinct on account of equation (27) which is verified by k_1 (and k_2). Therefore we are left with two independent sets of constants which we will, later on, express in terms of the a_{lm}, by means of the boundary conditions of our problem.

We can now integrate equations (24) and (25). It is easy to see that one gets

$$\Phi_p = \sum_{l,m} \Phi_p^{lm}(t) r^l P_l^m(\theta) e^{im\varphi} + \frac{1}{\varrho_p^0} \sum_{l,m,\omega_l} P_l^m(\theta) e^{im\varphi} e^{i\omega_l t} \cdot i\omega_l \left[\frac{D_{(\ )p}^{lm\omega_l}}{(k_1^{\omega_l})^2} j_l(k_1^{\omega_l} r) + \right. \tag{30}$$
$$\left. + \frac{D_{(2)p}^{lm\omega_l}}{(k_2^{\omega_l})^2} j_l(k_2^{\omega_l} r) \right]$$

(and a similar equation for Φ_n);

$$\frac{M\varepsilon_0}{e} \cdot V_i = \frac{C_p}{\alpha^2} - \frac{C_p}{\alpha^2} \cdot \frac{\sinh \alpha r}{\alpha r} + \sum_{l,m} V^{lm}(t) r^l P_l^m(\theta) e^{im\varphi} + \tag{31}$$
$$+ \sum_{l,m,\omega_l} P_l^m(\theta) e^{im\varphi} e^{i\omega_l t} \left[\frac{D_{(1)p}^{lm\omega_l}}{(k_1^{\omega_l})^2} j_l(k_1^{\omega_l} r) + \frac{D_{(2)p}^{lm\omega_l}}{(k_2^{\omega_l})^2} j_l(k_2^{\omega_l} r) \right],$$

$$\frac{M\varepsilon_0}{e} V_0 = \sum_{l,m} v^{lm}(t) r^{-(l+1)} P_l^m(\theta) e^{im\varphi}. \tag{32}$$

We can determine C_p, C_n, $\Phi_p^{lm}(t)$, $\Phi_n^{lm}(t)$, C_1, $V^{lm}(t)$, $v^{lm}(t)$ by imposing on the solution (28), (30), (31), (32), the following boundary conditions (as in WOESTE [6]):

1) The number of protons and neutrons must remain constant and equal to Z and N, respectively;

2) The normal components of the « surface velocities » $\boldsymbol{v}_p^\sigma$ and $\boldsymbol{v}_n^\sigma$ given by (30) must coincide with those calculated from $R(\theta, \varphi, t) = R_0\left[1 + \sum_{l,m} a_{lm}(t) P_l^m(\theta) e^{im\varphi}\right]$;

3) The scalar electromagnetic potential and the electric field must be continuous at the nuclear surface (the continuity of the normal component of the electric field meaning absence of a surface charge distribution); the potential outside the nucleus must, at great distances, go over into the potential due to a point charge Ze.

Denoting by $\bar{\varrho}_p$ and $\bar{\varrho}_n$ the average values of ϱ_p^{stat} and ϱ_n^{stat}, condition 1 gives

$$C_p = \bar{\varrho}_p \frac{X}{\sinh X}[X]; \qquad C_n = \bar{\varrho}\left[1 - \frac{2Z}{A} \frac{S}{S+Q}\right] \tag{33}$$

with

$$X = \alpha R_0; \qquad [X] = \frac{1}{3} \cdot \frac{X^2}{X \coth X - 1}; \qquad \bar{\varrho} = \bar{\varrho}_p + \bar{\varrho}_n;$$

and

$$\begin{cases} \bar{\varrho}_p[X] a_{00}(t) + \sum_{\omega_0} e^{i\omega_0 t} \left\{ \frac{D_{(1)p}^{00\omega_0}}{x_1^{\omega_0}} j_1(x_1^{\omega_0}) + \frac{D_{(2)p}^{00\omega_0}}{x_2^{\omega_0}} j_1(x_2^{\omega_0}) \right\} = 0, \\ \left\{ C_n - \frac{b}{a} \bar{\varrho}_p[X] \right\} a_{00}(t) + \sum_{\omega_0} e^{i\omega_0 t} \left\{ \frac{D_{(1)n}^{00\omega_0}}{x_1^{\omega_0}} j_1(x_1^{\omega_0}) + \frac{D_{(2)n}^{00\omega_0}}{x_2^{\omega_0}} j_1(x_2^{\omega_0}) \right\} = 0. \end{cases} \tag{34}$$

with

$$x_1^{\omega_l} = k_1^{\omega_l} R_0 ; \qquad x_2^{\omega_l} = k_2^{\omega_l} R_0 .$$

Condition 2 gives

$$(35) \qquad l R_0^{l-2} \Phi_p^{lm}(t) = \sum_{\omega_l} i\omega_l e^{i\omega_l t} \left\{ a_{lm\omega_l} - \frac{1}{\varrho_p^0} \left[\frac{D_{(1)p}^{lm\omega_l}}{x_1^{\omega_l}} j_l'(x_1^{\omega_l}) + \frac{D_{(2)p}^{lm\omega_l}}{x_2^{\omega_l}} j_l'(x_2^{\omega_l}) \right] \right\}$$

and a similar equation for $\Phi_n^{lm}(t)$.

For $l = 0$, as $j_0'(x) = -j_1(x)$, (35) gives equations similar to (34) but where the coefficients of $a_{00}(t)$ are ϱ_p^0 and ϱ_n^0. This shows that, for the system of partial differential equations (23), (24), (25) to have a solution satisfying the boundary conditions, 1), 2), 3), we must have

$$\varrho_p^0 = \bar{\varrho}_p[X] ; \qquad \varrho_n^0 = C_n - \frac{b}{a} \bar{\varrho}_p[X] ,$$

that is ϱ_p^0 and ϱ_n^0 must be the static proton and neutron distributions for $r = R_0$.

WOESTE [6], found convenient a similar choice in his treatment of the one fluid compressible liquid drop, although he does not state that this choice is a compatibility condition.

As to condition 3), the requirement on the potential at great distances gives

$$v^{00} = MZ/4\pi ;$$

the continuity of the potential at the nuclear surface gives

$$(36) \qquad \begin{cases} C_1 = \bar{\varrho}_p \left\{ \frac{1}{3} X^2 + [X] \right\} , \\ R_0^{-2} V^{00}(t) + \sum_{\omega_0} e^{i\omega_0 t} \left\{ \frac{D_{(1)p}^{00\omega_0}}{(x_1^{\omega_0})^2} j_0(x_1^{\omega_0}) + \frac{D_{(2)p}^{00\omega_0}}{(x_2^{\omega_0})^2} j_0(x_2^{\omega_0}) \right\} = 0 , \end{cases}$$

and for $l \geqslant 1$

$$(37) \qquad R_0^{l-2} V^{lm}(t) + \sum_{\omega_l} e^{i\omega_l t} \left\{ \frac{D_{(1)p}^{lm\omega_l}}{(x_1^{\omega_l})^2} j_l(x_1^{\omega_l}) + \frac{D_{(2)p}^{lm\omega_l}}{(x_2^{\omega_l})^2} j_l(x_2^{\omega_l}) \right\} = R_0^{-(l+3)} v^{lm}(t) ;$$

the continuity of E_r gives again the first equation of (34) and (for $l \geqslant 1$)

$$(38) \qquad -\varrho_p^0 a_{lm}(t) + l R_0^{l-2} V^{lm}(t) + \sum_{\omega_l} e^{i\omega_l t} \left\{ \frac{D_{(1)p}^{lm\omega_l}}{x_1^{\omega_l}} j_l'(x_1^{\omega_l}) + \frac{D_{(2)p}^{lm\omega_l}}{x_2^{\omega_l}} j_l'(x_2^{\omega_l}) \right. =$$

$$= -(l+1) R_0^{-(l+3)} v^{lm}(t) .$$

As was to be expected, continuity of E_φ and E_θ give again equation (37).

But, so far, we have only succeeded in expressing in terms of the a_{lm}'s, the D's with $l = m = 0$ (eqs. (29) and (34)). Besides (29), we still need two more distinct relations between the D's and a_{lm}'s. In our approximation, we can write immediately first integrals of the equations of motion (17), (18), corresponding to generalised Bernoulli equations:

$$(39)\qquad \begin{cases} \dfrac{\partial \Phi_p}{\partial t} + b\varrho_n + a\varrho_p + \dfrac{e}{M} V = F(t)\,, \\ \dfrac{\partial \Phi_n}{\partial t} + a\varrho_n + b\varrho_p = G(t)\,. \end{cases}$$

If we assume for the quantities in these equations the usual continuity properties (see e.g. LICHTENSTEIN [16]) they will also be valid on the nuclear surface. Before using these equations it is convenient to eliminate $r^{lm}(t)$ between (37) and (38); for $l \geqslant 1$

$$(40)\quad (2l+1)R_0^{l-2} V^{lm}(t) = \varrho_p^0 a^{lm}(t) \qquad \sum_{\omega_l} e^{i\omega_l t}\left\{\frac{D^{lm\omega_l}_{(1)p}}{x_1^{\omega_l}} j_{l-1}(x_1^{\omega_l}) + \frac{D^{lm\omega_l}_{(2)p}}{x_2^{\omega_l}} j_{l-1}(x_2^{\omega_l})\right\}.$$

Substituting (28), (30), (31) into (39), taking into account (33), (35), (36), (40) and making $r = R(\theta, \varphi)$, one gets, besides uninteresting relations between $F(t)$, $\dot{\Phi}_p^{00}(t)$ and $G(t)$, $\dot{\Phi}_n^{00}(t)$,

$$-R_0^2 a_{lm\omega_l}\left[\frac{\omega_l^2}{l} - \frac{d}{c_p(2l+1)}\right] + R_0^2 c_p\left[\frac{\omega_l^2}{l} - \frac{d}{c_p(2l+1)}\right]\cdot$$
$$\cdot\left\{\frac{D^{lm\omega_l}_{(1)p}}{x_1^{\omega_l}} j_{l+1}(x_1^{\omega_l}) + \frac{D^{lm\omega_l}_{(2)p}}{x_2^{\omega_l}} j_{l+1}(x_2^{\omega_l})\right\} + \left\{\left[aD^{lm\omega_l}_{(1)p} + bD^{lm\omega_l}_{(1)n}\right] j_l(x_1^{\omega_l}) + \right.$$
$$\left. + \left[aD^{lm\omega_l}_{(2)p} + bD^{lm\omega_l}_{(2)n}\right] j_l(x_2^{\omega_l})\right\} = 0\,;$$

$$-R_0^2 \frac{\omega_l^2}{l}\cdot a_{lm\omega_l} + R_0^2 c_n \frac{\omega_l^2}{l}\left\{\frac{D^{lm\omega_l}_{(1)n}}{x_1^{\omega_l}} j_{l+1}(x_1^{\omega_l}) + \frac{D^{lm\omega_l}_{(2)n}}{x_2^{\omega_l}} j_{l+1}(x_2^{\omega_l})\right\} +$$
$$+ \left\{\left[aD^{lm\omega_l}_{(1)n} + bD^{lm\omega_l}_{(1)p}\right] j_l(x_1^{\omega_l}) + \left[aD^{lm\omega_l}_{(2)n} + bD^{lm\omega_l}_{(2)p}\right] j_l(x_2^{\omega_l})\right\} = 0\,,$$

both valid for $l > 1$; taking into account (29) and simple relations between spherical Bessel functions of different orders, these equations can be simplified and rewritten as

$$(41)\qquad \left[-\frac{\omega_l^2}{l} + \frac{d}{c_p(2l+1)}\right] a_{lm\omega_l} + \frac{D^{lm\omega_l}_{(1)p}}{x_1^{\omega_l}}\left[\frac{d}{2l+1} j_{l-1}(x_1^{\omega_l}) - \omega_l^2 \frac{c_p}{l} j_l'(x_1^{\omega_l})\right] +$$
$$+ \frac{D^{lm\omega_l}_{(2)p}}{x_2^{\omega_l}}\left[\frac{d}{2l+1} j_{l-1}(x_2^{\omega_l}) - \omega_l^2 \frac{c_p}{l} j_l'(x_2^{\omega_l})\right] = 0\,,$$

$$a_{l m \omega_l} + c_n \left[\frac{D^{l m \omega_l}_{(1)n}}{x^{\omega_l}_1} j'_l(x^{\omega_l}_1) + \frac{D^{l m \omega_l}_{(2)n}}{x^{\omega_l}_2} j'_l(x^{\omega_l}_2) \right] = 0 \,. \tag{42}$$

Equations (29), (41), (42) give $D^{l m \omega_l}_{(1)p}$, $D^{l m \omega_l}_{(2)p}$, $D^{l m \omega_l}_{(1)n}$, $D^{l m \omega_l}_{(2)n}$ in terms of $a_{l m \omega_l}$, for $l \geqslant 1$; for $l = 0$ one has, instead of (41), (42), equations (34).

For $l \geqslant 1$ one gets

$$\begin{cases} D^{l m \omega_l}_{(1)n} = x^{\omega_l}_1 Z^{l \omega_l}_1 \cdot a_{l m \omega_l}, \\ D^{l m \omega_l}_{(1)p} = - \left[\dfrac{a}{b} + \dfrac{c_n}{b} \cdot \dfrac{\omega_l^2}{(k^{\omega_l}_1)^2} \right] D^{l m \omega_l}_{(1)n} \end{cases} \tag{43}$$

with

$$\begin{aligned} Z^{l w_l}_1 \left\{ \left[\frac{a}{b} + \frac{c_n}{b} \frac{\omega_l^2}{(k^{\omega_l}_1)^2} \right] \left[\frac{c_p}{l} \omega_l^2 j'_l(x^{\omega_l}_1) - \frac{d}{2l+1} j_{l-1}(x^{\omega_l}_1) \right] - \right. \\ \left. - \frac{j'_l(x^{\omega_l}_1)}{j'_l(x^{\omega_l}_2)} \left[\frac{a}{b} + \frac{c_n}{b} \frac{\omega_l^2}{(k^{\omega_l}_2)^2} \right] \left[\frac{c_p}{l} \omega_l^2 j'_l(x^{\omega_l}_2) - \frac{d}{2l+1} j_{l-1}(x^{\omega_l}_2) \right] \right\} = \\ = \frac{\omega_l^2}{l} - \frac{d}{c_p(2l+1)} + \frac{1}{c_n j'_l(x^{\omega_l}_2)} \left[\frac{a}{b} + \frac{c_n}{b} \frac{\omega_l^2}{(k^{\omega_l}_2)^2} \right] \left[\frac{c_p}{l} \omega_l^2 j'_l(x^{\omega_l}_2) - \frac{d}{2l+1} j_{l-1}(x^{\omega_l}_2) \right] \end{aligned} \tag{44}$$

and similar expressions for $D^{l m \omega_l}_{(2)p}$ and $D^{l m \omega_l}_{(2)n}$, which can be obtained from (43) and (44) by simply interchanging the indices (1) and (2).

The eigenfrequencies can now be determined if one takes into account (14). That equation gives

$$\varepsilon_s - n_0 P - \tfrac{1}{2}(Q\varphi_\sigma^2 + S\psi_\sigma^2) - n_0 Q \varphi_\sigma + \gamma C_m = 0 \,,$$

from which one can derive, in our approximation,

$$\gamma \delta C_m = \varrho_n^0 [a \delta \varrho_n^\sigma + b \delta \varrho_p^\sigma] + \varrho_p^0 [b \delta \varrho_n^\sigma + a \delta \varrho_p^\sigma] \,, \tag{45}$$

where the δ's mean deviations from the static (ground state) values.

A rather long, but otherwise straightforward calculation, gives for the surface

$$R = R_0 [1 + \sum_{l,m} a_{lm}(t) P_l^m(\theta) e^{im\varphi}]$$

a mean curvature

$$C_m(\theta, \varphi, t) = \frac{2}{R_0} + \sum_{l,m} \frac{1}{R_0} (l-1)(l+2) a_{lm}(t) P_l^m(\theta) e^{im\varphi}$$

which reduces to the expression given by WOESTE [6], for a cylindrically symmetric surface.

Therefore

$$\delta C_m(\theta,\varphi)=\frac{1}{R_0}\sum_{l,m}(l-1)(l+2)a_{lm}(t)P_l^m(\theta)e^{im\varphi}.$$

Using $\delta\varrho_p^\sigma$, $\delta\varrho_n^\sigma$ given by (28) one finally gets the eigenfrequencies equation (for $l\geqslant 1$)

$$(46)\quad \frac{\gamma}{R_0^3}(l-1)(l+2)-\frac{a^2-b^2}{ac_p^2R_0^2}(X\coth X-1)=\frac{1}{x_1^{\omega_l}}j_l(x^{\omega_l})Z_1^{l\omega_l}\left\{\omega_l^2-(\omega_l^2-d/c_p)\cdot\left[\frac{a}{b}+\frac{c_n}{b}\frac{\omega_l^2}{(k_1^{\omega_l})^2}\right]\right\}+\frac{1}{x_2^{\omega_l}}j_l(x_2^{\omega_l})Z_2^{l\omega_l}\left\{\omega_l^2-(\omega_l^2-d/c_p)\left[\frac{a}{b}+\frac{c_n}{b}\frac{\omega_l^2}{(k_2^{\omega_l})^2}\right]\right\}.$$

For $l=0$ the eigenfrequencies equation could be obtained from (45), (29) and (34). From (46) one can see that the eigenfrequencies ω_l do not depend on m, as might be expected.

3. – For $l=1$ the eigenfrequencies equation can be written

$$F(\omega)=-\frac{a^2-b^2}{ac_p^2R_0^2}(X\coth X-1),$$

with

$$F(\omega)=\frac{1}{x_1}j_1(x_1)Z_1[\omega^2-(\omega^2-d/c_p)I_1]+\frac{1}{x_2}j_1(x_2)Z_2[\omega^2-(\omega^2-d/c_p)I_2],$$

$$2(b^2-a^2)k^2=a[(c_n+c_p)\omega^2-d]\pm\{a^2[(c_n-c_p)\omega^2+d]^2-4b^2c_n\omega^2[d-c_p\omega^2]\}^{\frac{1}{2}},$$

$$I_n=\frac{a}{b}+\frac{c_n}{b}\frac{\omega^2}{k_n^2};\qquad H_n=c_p\omega^2j_1'(x_n)-(d/3)j_0(x_n),\qquad n=1,2$$

$$\beta Z_1=(\omega^2-(d/3c_p))j_1'(x_2)+(1/c_n)I_2H_2,$$

$$\beta Z_2=-(\omega^2-(d/3c_p))j_1'(x_1)-(1/c_n)I_1H_1,$$

$$\beta=I_1H_1j_1'(x_2)-I_2H_2j_1'(x_1).$$

The curves $y=F(\omega)$ are very similar to the ones given by WOESTE [6], and the lowest eigenfrequencies for $l=1$ were determined by calculating a large enough number of points of such curves.

We used for S and r_0 the same numerical values as JENSEN-STEINWEDEL [3] and DANOS [4]:

$S=29.40\cdot10^{-43}$ erg·cm^3 (corresponding to a term $20(N-Z)^2A^{-1}$ MeV

$r_0=1.42\cdot10^{-13}$ cm in the semi-empirical mass formula)

and for M the nucleon rest mass.

For Q we used

$$Q = 8.29 \cdot 10^{-43} \text{ erg} \cdot \text{cm}^3,$$

corresponding to a « sound velocity » $u = 0.11c$ (as in FLÜGGE and WOESTE [5]).

Comparing this value with the estimates of BETHE [17] and PRESENT [18], FEENBERG [19] and SWIATECKI [13] — $0.085c$; $0.07 - 0.11c$; $0.085 - 0.09c$ respectively — one sees that it corresponds to a rather small compressibility.

In the following table we compare our results for five nuclei (^{63}Cu, ^{107}Ag, ^{181}Ta, ^{197}Au, ^{232}Th) with the experimental data, as well as with the results obtained by JENSEN, STEINWEDEL and by DANOS. Jensen-Steinwedel's results are calculated by the formula $\varepsilon = 60A^{-\frac{1}{3}}\sqrt{4NZ/A^2}$ and Danos' results are obtained from those values by adding a correction which DANOS expresses as $\Delta\omega/\omega_0 = 1.16 \cdot 10^{-4} A^{1.3}$.

Nucleus	JENSEN-STEINWEDEL	DANOS	Our calculations	Experimental values
^{63}Cu	15.1 MeV	15.4 MeV	17.6 MeV	17.3 – 18.1 MeV
^{107}Ag	12.6	13.2	15.2	16.3
^{181}Ta	10.6	11.6	13.3	13.9 – 15.1
^{197}Au	10.3	11.4	13.1	13.9 – 15.0
^{232}Th	9.8	11.1	12.6	13.8 – 15.0

This expression seems, however, to overestimate the actual correction. The experimental data are taken from the review article by STRAUCH [20] and from the literature (papers later than February 1952). Some results seem to be arrived at by the study of only some of the possible photonuclear reactions; and, in some cases, the numbers given probably do not represent the resonance energies for photon absorption, because photonuclear reactions of low enough threshold have not been considered. Fig. 1 is a graphical representation of the results contained in the table.

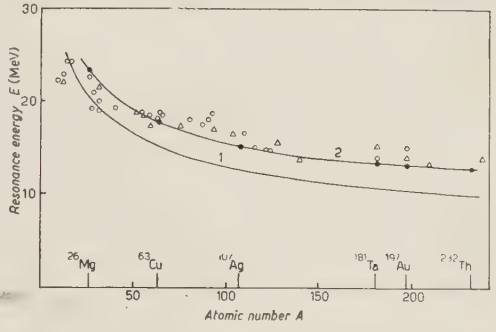

Fig. 1. – Resonance energies for dipole photon absorption. The theoretical curves 1 and 2 correspond to Jensen-Steinwedel's results and to the results taking account of compressibility. Danos' results are not represented. Only the latest experimental data are included: ○ MONTALBETTI, KATZ and GOLDEMBERG [21]; △ NATHANS and HALPERN [22].

In the table we have given only the limits within which the different experimental values lie. The actual experimental values are the following (* means that the value for the nucleus refered to has not been measured but the authors give curves from which the quoted value was obtained):

^{63}Cu: 17.5, 17.5, 18.1 [20]; 18.1 [21]; 17.3* [22]; 17.5 [23]; 17.3 [24].

^{107}Ag: 16.3 [21]; 16.3^{*}_{5} [22].

^{181}Ta: 13.9, 13.9 [20]; 14.2 [23]; 14.6 [25]; 14.2 [26]; 14.1 [27]; 15.0 [28]; 15.1 [22].

^{197}Au: 15.0 [29]; 13.9 [22].

^{232}Th: 13.8* [21]; 13.8* [22]; 15.0 (+) [30].

It is also interesting to compare the mass number dependences of the resonance energy for nuclear photon absorption given by the different theories. JENSEN and STEINWEDEL get an $A^{-\frac{1}{3}}$ dependence (in fact stil more pronounced due to the increase of neutron excess with mass number). Experimentally, STRAUCH [20], MONTALBETTI, KATZ and GOLDEMBERG [21], NATHANS and HALPERN [22], give $A^{-0.22}$, $A^{-0.186}$, $A^{-0.19}$, respectively. Our results correspond to an $A^{-0.26}$ mass number dependence, in the medium and heavy nuclei regions (Fig. 2).

A change in the value of the constants S and r_0, such as that suggested by the latest analysis of the semi-empirical mass formula by GREEN and ENGLER [31], would make Jensen-Steinwedel's numerical results greater; but still it would not alter the $\sim A^{-0.34}$ dependence on the mass number which

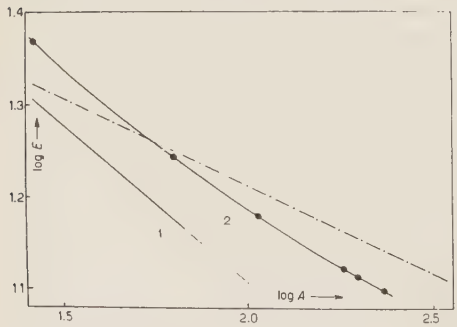

Fig. 2. – Mass number dependence of the resonance energies for photon absorption. Curve 1 corresponds to Jensen-Steinwedel's results neglecting neutron excess; curve 2 was drawn so as to fit our numerical results for a few stable nuclei. The experimental curve corresponds to the results for photonuclear reactions given by MONTALBETTI, KATZ and GOLDEMBERG [21] and NATHANS and HALPERN [22]. These results are not directly comparable to the theoretical curves except for rather heavy nuclei; for light nuclei the consideration of the (γ, p) reactions would probably give a more marked dependence of $\log E$ on $\log A$.

(+) This value corresponds to photofission while the other two refer to « photoneutron reactions » — $(\gamma, n)+(\gamma, 2n)+(\gamma, n+p)$.

they obtain (*). The same can be said concerning the theoretical treatments of the photonuclear effect recently published by FERENTZ, GELL-MANN and PINES [10] and by REIFMAN [9]. The same change in the constants S and r_0 would also make our results increase. For instance, with $r_0 = 1.2 \cdot 10^{-13}$ cm instead of $1.42 \cdot 10^{-13}$ cm (and $S = 18.53$ erg·cm^3, $Q = 5.23_5$ erg·cm^3, corresponding to the new value of r_0 but to the same asymmetry energy term in the semi-empirical mass formula and the same compressibility) the resonance energy for ^{181}Ta would be 16.2 MeV instead of 13.3 MeV. It seems, therefore, that compressibility of the « nuclear matter » should not be neglected in the treatment of photon absorption in the nuclear photoeffect. The importance of the compressibility for ground state or low-lying excited state nuclear properties must obviously be less. It would be possible to calculate the Hamiltonian for a compressible two-fluid liquid drop and compare the phonon energies, the parameters B_2 and C_2 (in A. BOHR's notation), etc., with those used by BOHR and MOTTELSON (which correspond to an incompressible, uniformly-charged liquid drop); however, we will not tackle this problem now.

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(*) This is also true if one changes the values of M, as suggested by DANOS [4], considering it as an « effective mass » smaller than the nucleon mass.

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RIASSUNTO (*)

Si formula un modello nucleare generale a due fluidi per il caso in cui la densità dell'energia nucleare dipende soltanto dai valori locali delle densità nucleoniche e non dalle loro derivate. Si derivano le equazioni idrodinamiche assieme ad un'equazione di Laplace generalizzata tenendo conto della compressibilità. Per qualche nucleo si calcolano le energie corrispondenti alle più basse oscillazioni di dipolo e si confrontano coi risultati di Danos e Steinwedel, per le energie di risonanza per l'assorbimento fotonico. Come era prevedibile, il miglior accordo coi dati sperimentali si ottiene pei nuclei di peso medio.

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Optical Model of Nuclei and Elastic Backscattering of π-Mesons.

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Summary. — A general inequality is established connecting the differential elastic cross-section at 90° and 180° in the limiting case of a black nucleus. The optical model of nuclei is applied to the interaction of π-mesons with the heavy elements of Ilford G.5 plates.

1. – Introduction.

If we describe the nuclei with the optical model ([1,2]), in every nuclear reaction the incoherent events manifest themselves in two manners which are experimentally distinct and independent: they appear directly in the reaction cross-section and indirectly in the elastic cross-section as shadow scattering. In the elastic cross-section the shadow scattering appears superimposed to the diffraction scattering and to the terms which contain at once the real and the imaginary phase-shifts. The separation of the various terms of the elastic cross-section is only exceptionally possible and, consequently, it is very difficult to get informations about the incoherent events from the examination of the elastic ones. If the particles involved in the nuclear reaction are charged, the shadow scattering is generally completely hidden by the diffraction scattering. In this paper we will examine the particular details of the shadow scattering and we will conclude with a rule valid for black nuclei, according to which, the incoherent events produce an increase of the elastic cross-section in the interval 90°-180°, beyond the large effect at $\theta \sim \lambda\!\!\!^{-}/R$ ([3]). Then, if the potential responsible of the diffraction does not give rise to an increase of the cross-section

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in the interval 90°-180°, (we exclude for instance spin-orbit coupling), we must expect that, even in the case of nuclei with a transparency different from zero, the main contribution to the elastic cross-section in the interval 90°-180° is due to shadow scattering. We will apply this model to the scattering of π-mesons by heavy nuclei of Ilford G.5 plates, assuming different nuclear transparencies.

2. – Black Nucleus.

The differential elastic cross-section for a black nucleus is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2k^2} \left| \sum_0^{l_{\max}} (2l+1) P_l(\theta) \right|^2 ,$$

with $l_{\max}$ depending from the range of the nucleus. The recurrence formula

$$(l+1)P_{l+1} = (2l+1)\cos\theta\, P_l - lP_{l-1}$$

yelds

$$P_l(\pi) = (-1)^l$$

and

$$P_{2l+1}\left(\frac{\pi}{2}\right) = 0 ; \qquad P_{2l+2}\left(\frac{\pi}{2}\right) = (-1)^{l+1} \frac{2l+1}{2l+2} \cdot \frac{2l-1}{2l} \cdots \frac{3}{4} \cdot \frac{1}{2}$$

Then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\pi) = \frac{1}{2k^2} \left| \sum_0^{l_{\max}} (2l+1)(-1)^l \right|^2 = \frac{1}{2k^2} (l_{\max}+1)^2 ,$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\frac{\pi}{2}\right) = \frac{1}{2k^2} \left| 1 + \sum_{l_{\text{even}}>0}^{l_{\max}} (2l+1)(-1)^{l/2} \frac{l-1}{l} \cdot \frac{l-3}{l-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \right|^2 . \tag{1}$$

The terms of the sum (1) have alternative signs and their absolute value is increasing with l. We suppose that $l_{\max}$ and $l_{\max}/2$ are even. By summing every positive term in (1) with the preceding negative one, we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\frac{\pi}{2}\right) = \frac{1}{2k^2} \left| 1 + \sum_{l_{\text{even}}>0,\,(l/2)_{\text{even}}}^{l_{\max}} \frac{2l-1}{2l} \cdot \frac{l-3}{l-2} \cdot \frac{l-5}{l-4} \cdots \frac{3}{4} \right|^2 \tag{2}$$

and every term of the sum (2) is positive, < 1 and always decreasing with l.

Then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\frac{\pi}{2}\right) \leqslant \frac{1}{2k^2}\left(1 + \frac{l_{\max}}{4}\right)^2 ,$$

and at last

$$\frac{\mathrm{d}\sigma(\pi)}{\mathrm{d}\sigma(\pi/2)} \geqslant 16\left(\frac{1 + l_{\max}}{4 + l_{\max}}\right)^2 \geqslant 1,$$

the sign of equality being valid only for $l_{\max} = 0$.

In the case of $l_{\max}$ even and $l_{\max}/2$ odd it is ,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\frac{\pi}{2}\right) \leqslant \frac{1}{2k^2}\left(\frac{l_{\max} + 4}{4}\right)^2$$

and at last

$$\frac{\mathrm{d}\sigma(\pi)}{\mathrm{d}\sigma(\pi/2)} \geqslant \frac{(l_{\max} + 1)^2}{[(l_{\max}/4) + 1]^2} > 1 , \tag{3}$$

In the case of $l_{\max}$ odd, we obtain « a fortiori » the inequality (3). For getting a quantitative sample of the ratio of the cross-sections we observe that, for $l_{\max} = 3$, it is $\mathrm{d}\sigma(\pi)/\mathrm{d}\sigma(\pi/2) \sim 7$.

Total elastic cross-section. – After the consideration of the differential cross-section, it is useful to know the ratio of the total cross-sections between (0°-90°) and (90°-180°). In Table I are collected the results for some $l_{\max}$, always in the case of a black nucleus.

TABLE I.

$l_{\max}$	0	1	2	3
$\dfrac{\int_0^{\pi/2} (\mathrm{d}\sigma/\mathrm{d}\Omega)_{\mathrm{el.}}\,\mathrm{d}\Omega}{\int_{\pi/2}^{\pi} (\mathrm{d}\sigma/\mathrm{d}\Omega)_{\mathrm{el.}}\,\mathrm{d}\Omega}$	50%	14%	12%	7,5%

As soon as $l_{\max}$ is larger than few unities, the total backscattering becomes some percents of the forward scattering, but contemporaneously the differential elastic cross-section around 180° is more and more increasing.

3. – π^+-Mesons of 62 MeV on Nuclei of Ilford G.5 Plates.

According to experimental results ([4]), ([5]), ([6]), the angular distribution of the coherent events gives a characteristic backscattering and, according to the latest results ([6]), the group of the backscattered mesons is surely an elastic one, independently of any arbitrary energy cut-off. The chemical composition of Ilford G.5 plates can be schematised as made up with two types of nuclear elements; i.e. «light elements» represented by $A = 14$, $Z = 7$, with a concentration of 57%, and «heavy elements» represented by $A = 95$, $Z = 42$, with a concentration of 43%. We have applied the model only to the heavy elements, the use of a statistical model being doubtful when the number of nucleons is of the order of 10.

We assume the nucleus to be limited by a sharp boundary $R = 1{,}4 \cdot 10^{-13} A^{\frac{1}{3}}$; naturally the case of a boundary sharply defined is not the most physically interesting. We have calculated the rigorous solution up to $l = 4$ (*) in order to compare it with the W.K.B.J. one; but we must choose the W.K.B.J. solution as the most physical ([1]).

Assuming a constant nuclear density, the potential experienced by π^+-mesons in the «inner nuclear region» is

$$\frac{Ze^2}{R}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right] - (V + i\Phi)$$

and in the «outer region» it is the Coulomb potential.

The inner non relativistic solution is

$$\Psi = \sum_l R_l(kr) P_l(\theta) = \sum_l F_a(\alpha_l, \gamma_l, \varepsilon r^2) r^{\gamma_l - \frac{3}{2}} \exp\left[-\varepsilon r^2/2\right] P_l(\theta) .$$

The boundary conditions of R_l allow the univocal determination of the coefficients appearing in the r.h.s. member.

Assuming an attractive potential V and an imaginary potential Φ, both of 10 MeV depth, (corresponding to a mean free path in nuclear matter of $\sim 6 r_0$), the confluent hypergeometric function F_a can be calculated with a series development in increasing powers of εr^2, very rapidly converging. The complex phase-shifts δ_l are obtained by matching the solutions relative to the

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([4]) G. Bernardini, E. T. Booth and L. Ledermann: *Phys. Rev.*, **83**, 1277 (1951). G. Bernardini and F. Lévy: *Phys. Rev.*, **84**, 610 (1952).

([5]) H. Byfield, J. Kessler and L. Ledermann: *Phys. Rev.*, **86**, 17 (1952).

([6]) A. Minguzzi, G. Puppi and A. Ranzi: *Nuovo Cimento*, **11**, 697 (1954).

(*) In this calculation $kR = 4.30$ but we have limited to $l = 4$ for missing tabulated values of Coulomb functions for larger angular momenta.

two regions across the nuclear boundary

$$\operatorname{tg}\,\delta_l = \frac{\dot{F}_l - (1/k)\,[\dot{\chi}_l/\chi_l]\,F_l}{(1/k)\,[\dot{\chi}_l/\chi_l]\,G_l - \dot{G}_l}$$

with F_l and G_l respectively the regular and irregular Coulomb function [7] and $\chi_l = R_l r$.

The phase-shifts, in W.K.B.J. approximation, have been calculated with the reduced formola of Fernbach [8]

$$\delta_l = \lim_{\varrho\to\infty}\left\{-\frac{1}{2}\int_{l+\frac{1}{2}}^{\varrho}\left[\varrho^2 - \left(l+\frac{1}{2}\right)^2\right]^{-\frac{1}{2}} U(r)\varrho\,\mathrm{d}\varrho\right\},$$

where $\varrho = kr$, and $U(r)$ is $2\mu/\hbar^2k^2$ times the potential. In our case the « classical waves » reduce to the first four. The asymptotic wave function is

$$\Psi_{r\to\infty} \to \frac{1}{\varrho}\sum_0^\infty i^l\,(2l+1)\,P_l(\theta)\,\exp[i(\eta_l+\delta_l)]\,\sin\left(\varrho - \frac{1}{2}\,l\pi - n\ln 2\varrho + \eta_l + \delta_l\right);$$

the reaction cross-section and the differential elastic cross-section are respectively:

$$\sigma_r = \frac{4\pi}{k^2}\sum_0^\infty (2l+1)\,\frac{\mathrm{I.P.}\,(\operatorname{tg}\,\delta_l)}{1+|\operatorname{tg}\,\delta_l|^2 + 2\mathrm{I.P.}\,(\operatorname{tg}\,\delta_l)},$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{el.} = \frac{1}{k^2}\left|-\frac{n}{2\sin^2\theta/2}\exp\left[-in\ln\sin^2(\theta/2) + 2i\eta_0\right] + \right.$$

$$\left. + \sum_0^\infty (2l+1)\,P_l(\theta)\,\exp\left[i(2\eta_l+\delta_l)\right]\sin\delta_l\right|^2.$$

The differential elastic cross-sections are collected in Fig. 1. The exact and W.K.B.J. solutions lead to similar results; we see clearly that the increase of the scattering cross-section is kept even in the case of a nuclear transparency different from zero. The reaction cross-sections are sensibly different in the two cases and larger in W.K.B.J. approximation. In Table II, columns (*a*), (*b*),

(7) I. Bloch, M. H. Hull Jr., A. A. Broyles, W. G. Bouricius, B. F. Freeman and G. Breit: *Rev. Mod. Phys.*, **23**, 147 (1951).

(8) K. M. Gatha and R. J. Riddel Jr.: *Phy . Rev.*, **86**, 1035 (1952).

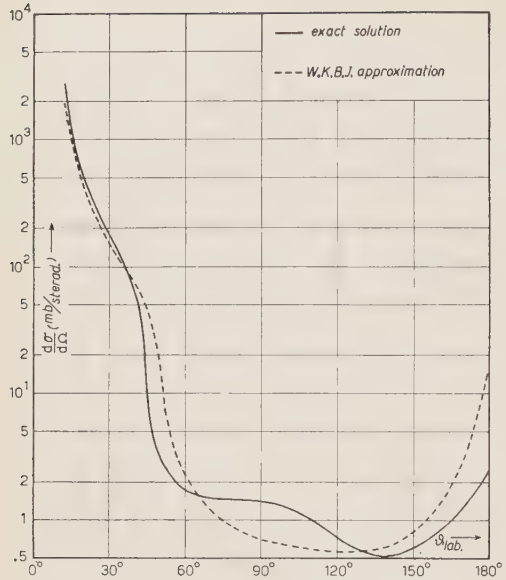

Fig. 1.

are collected the integral results for the exact and W.K.B.J. solutions respectively. The Born approximation gives a result which does not differ from the exact one up to 70°. From 130° to 180° it shows a small increase, but in this region it remains smaller by a factor 10 in respect to the exact one.

Since the « heavy elements » contribute only for 43% to the total cross-sections, we see that the theoretical results are far from the experimental ones; then it is necessary to increase the scattering and reactioncros s-sections, introducing a deeper potential well. This will be done in the next section.

4. – Relativistic Calculation for π-Mesons of 62 MeV in W.K.B.J. Approximation.

Starting from the relativistic equation

$$(E - W)^2\Psi = (c^2p^2 + \mu^2c^4)\Psi\,,$$

we substitute [8]

$$(E - W)^2 - \mu^2c^4 = (\hbar c k_c)^2$$

with

$$k_c = k + k_2 + \tfrac{1}{2}\,i\Gamma,$$

where

$$k = \frac{(E^2 - \mu^2c^4)^{\frac{1}{2}}}{\hbar c}\,, \qquad k + k_2 = \frac{[(E - W)^2 - \mu^2c^4)]^{\frac{1}{2}}}{\hbar c}\,.$$

W is the potential experienced by the mesons and $1/\Gamma$ is the mean free path of the π-mesons in nuclear matter. We make the simplest hypothesis that the only real potential acting on the π-mesons is the Coulomb potential in the « outer region » and the Thomson potential in the « inner region ». For the mean free path of the π^-, in nuclear matter, we must expect a shorter mean free path than for the π^+, because, owing to the Coulomb attraction, the π^--mesons are impinging on the nucleons with larger energy than the π^+, and the

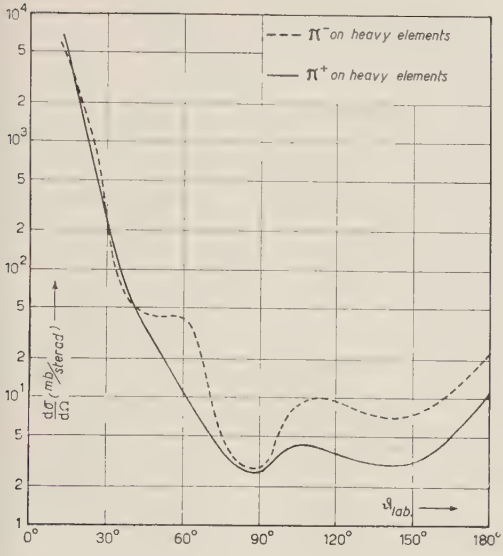

Fig. 2.

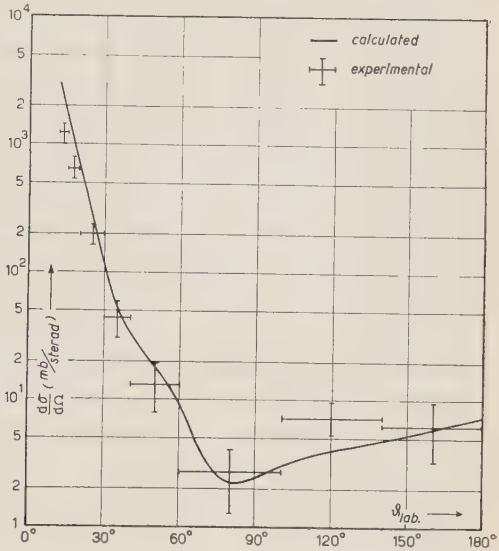

Fig. 3.

elementary cross-sections of scattering, charge exchange and true absorption increase with energy; moreover the larger is the energy of the incident meson the smaller is the cut-effect of the Pauli principle. The difference between the two mean free paths being hard to estimate, we have chosen for the π^+ as well as for the π^-

$$1/l' = 3r_0 .$$

This is equivalent to an imaginary potential well of ~ 20 MeV depth. In Fig. 2 have been reported the results for the heavy elements. In Fig. 3 the comparisons have been given between the experimental results of the π^+ in photographic plates ([5]) and the « weighted results » obtained assuming for the light elements the experimental results given by LEDERMANN and COW. in C ([5]) and for the heavy elements the latest theoretical results (Fig. 2).

In Table II are collected the integral results. The columns (*c*), (*d*) are referring to π^+ and π^- respectively and the column (*e*) refers to « weighted results ».

TABLE II.

	a	*b*	*c*	*d*	*e*
$\int_{12^\circ}^{180^\circ} (d\sigma/d\Omega)_{el.}\, d\Omega$ (mb.)	430	530	1 250	1 330	640
$\sigma_{reaction}$ (mb.)	450	740	1 080	1 180	680

5. – Conclusion.

At the energy of 62 MeV the experimental results of the π^+ mesons are not too far from the theoretical ones, but the latter show a depression between 90°-130° which is very much sensitive on the total elastic backscattering cross-section. On the other hand the experimental results are not statistically sufficient to allow an exact separation of the coherent events from the incoherent ones for angles larger than 90°.

It may be that more refined experimental results can be accounted with the introduction of a real potential, near the imaginary one; but it is out of doubt, from the general considerations, that the incoherent events produce an increase of the elastic cross-section in the interval 90°-180°.

It is interesting to see that, assuming the same nuclear potential for π^+ and π^--mesons, the relativistic equation gives automatically the correct inequality between the total cross-sections of the two types of mesons.

I should like to express my indebtness to Prof. G. Puppi for valuable discussions and advices, and my thanks to Miss A. Vignudelli for computational assistance.

RIASSUNTO

Viene ricavata una disuguaglianza generale che lega la sezione d'urto differenziale elastica a 90° e 180° nel caso limite del nucleo nero. Il modello ottico del nucleo è usato per l'interazione di mesoni π coi nuclei pesanti delle lastre fotografiche Ilford G.5.

A Five Particle Decay of a Heavy Meson?

G. S. Shrikantia

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(ricevuto il 15 Settembre 1954)

An unusual event has been observed in a stack of 600 μ G5 stripped emulsions exposed in a balloon flight of the 1953 Sardinian expedition. A facsimile drawing of the event is shown in the figure. This can be interpreted as a decay of a heavy meson (track No. 1) into 5 singly charged particles (tracks 2-6). This interpretation is not absolutely sure but is given here on account of the similarity to the cloud-chamber event observed by the Princeton Group (Hodson *et al.*, 1954) (1).

Because of the geometry of the event, the only track suitable for measurement is track No. 2 of blob density $1.05 \pm .04$ times plateau and $p\beta = 258 \pm 47$ MeV/c indicating the particle was an electron or a light meson. The remaining four tracks dip too steeply for reliable scattering measurements but the blob densities are found to be close to the plateau value.

The interpretation of the event rests on the identification of particle No. 1

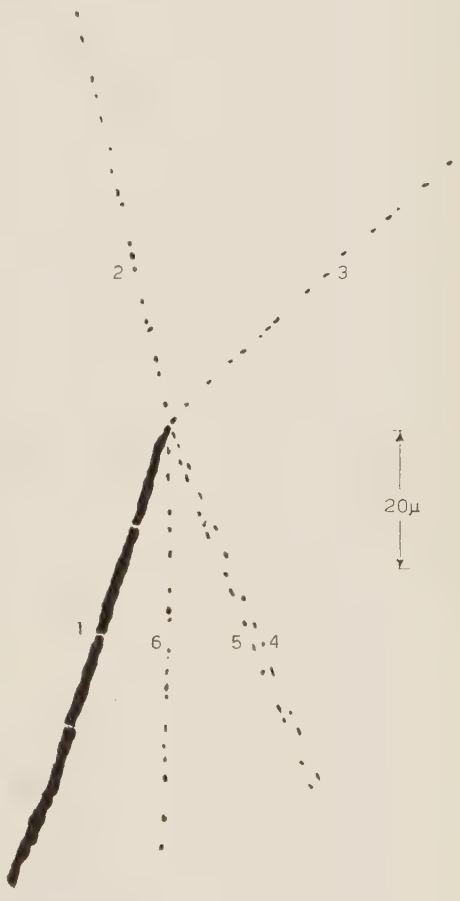

Fig. 1.

(*) Also supported by the Nuclear Research Foundation within the University of Sydney.

(1) A. L. Hodson, W. H. Arnold, J. Ballam, D. R. Harris, R. R. Rau, G. T. Reynolds, H. Gursky, J. Hardy, W. Hooke e G. K. Lindeberg: Private Communication.

as a K-particle moving towards the event. The track of this particle dips at an angle of 45° and it is therefore necessary to trace it through a number of emulsion sheets in order to carry out measurements. The tracing was carried out through 10 successive sheets. Due to emulsion distortion and a high track density, an uncertainty exists as to whether the same track has been followed in all the sheets.

Assuming the tracing through to be correct, then particle No. 1 was moving towards the event, and its track had a grain density of (3.2 ± 0.2) times the plateau value at a residual range of 8400 μ, indicating that it was probably due to a K-particle.

Given that the mass of particle 1 is 1000 m_e, then not more than one of the secondary particles can be a light meson on momentum and energy conservation grounds. The event seems to be similar to the one recently observed at Princeton, the most plausible interpretation being that a heavy meson decays into a π-meson and 4 electrons.

I am grateful to Drs. V. D. Hopper and E. P. George for useful discussion.

On the Positive Excess of Mesons of High Energy.

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Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 21 Settembre 1954)

At the conclusion of a long series of experiments on the positive excess of μ-mesons carried out in this Laboratory in recent years, we have made a number of differential measurements of the positive excess in the energy interval from 1 to 21 GeV. Measurements were made in the galleries of the electricity generating station at Agordo (Belluno) at 600 m above sea level, under various thicknesses of rock including zero thickness. The width and the mean energy of the energy bands on which measurements of the excess were made, are determined by the characteristics of the experimental layout, and by the cut-off produced by the different thicknesses of rock under which the apparatus was put. Measurements were made with two sets of apparatus. With the first arrangement it was possible to make measurements up to 10 GeV, the results of which have already been published in a letter by one of us in collaboration with other workers [1]. Having made this measurements it was thought to be highly desirable to have more information about the value of the excess at high energies, and the apparatus was redesigned accordingly. Since the new apparatus had higher counting rate, it was possible to obtain a greater statistical precision

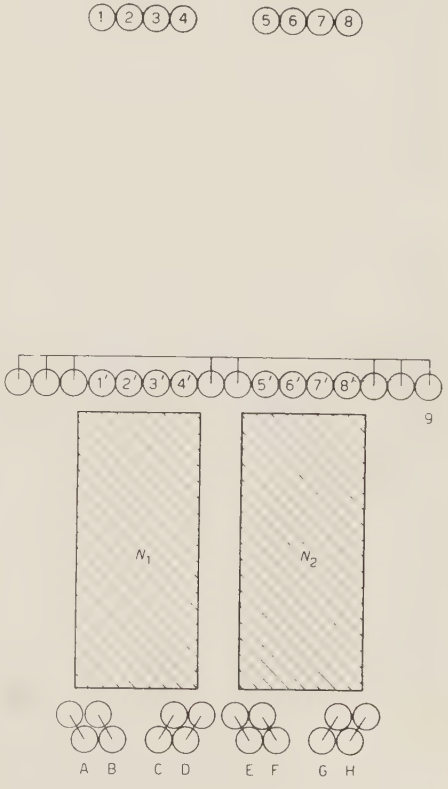

Fig. 1.

[1] E. Beretta, I. Filosofo, G. Puppi and B. Sommiacal: *Nuovo Cimento*, **10**, 1354 (1953).

in the results already obtained, and to extend the measurements to include a narrow band with its mean value at 21 GeV.

This later version of the apparatus is shown in Fig. 1. The pole-pieces N_1, N_2 having a magnetic field parallel to the axes of the counters, separate the positive and negative μ-mesons, which are then counted by the following system of counters. There are 16 counting channels, each consisting of 3 counters in triple coincidence. They may be divided into two groups of 8 channels according to the geometry; one group of the type 2-2′-D or 4-4′-B which count incoming particles lying in a solid angle with a vertical axis; the other group of the type 3-2′-C or 1-2′-A which count incoming particles lying in solid angles inclined on either side of the vertical when viewed in the direction of the magnetic field. Other than the different mean directions of acceptance of the particles recorded, the two groups of channels have the same efficiency of separation of particles lying in identical energy bands. The three counters of each channel are arranged in such a way that the « collector » counter, i.e. the bottom one, lies somewhat outside the solid angle defined by the other two, and in this manner particles of a selected sign which have been deflected by the magnetic field are counted. It is very improbable that particles of the other sign which have been deflected by multiple Coulomb scattering will also be counted. It therefore results that this particular experimental arrangement has a very high efficiency of separation and of counting the positive and negative penetrating particles. For a more complete discussion of the apparatus we would refer to a previous paper (2).

The magnetic fields in the two pole-pieces have opposite senses, both being parallel to the axes of the counters. The frequency of the particles of one sign has been obtained by counting the triple coincidences:

1-1′-C; 2-2′-D; 7-7′-E; 8-8′-F;

3-2′-C; 4-3′-D; 5-6′-E; 6-7′-F;

and that of the opposite sign by counting coincidences:

3-3′-A; 4-4′-B; 5-5′-G; 6-6′-H;

1-2′-A; 2-3′-B; 7-6′-G; 8-7′-H.

Every 24 hours the direction of the field in the pole-pieces was reversed so as to eliminate sistematic errors due to asymmetry of the geometry or of the efficiency of the apparatus.

With the intention of reducing the background due to showers and casual coincidences, the lower part of the apparatus was screened with lead, and anticoincidence sets were incorporated. The triple coincidences which are normally counted are in anticoincidence with any double coincidence produced between any two counters 1′2′3′...8′9. Anticoincidences are also produced by double coincidences between any one counter ABGH and any one counter CDEF. With these precautions the background when measured without a magnetic field was only a few percent of the counting rate when operating under normal conditions with a field.

The counters were of the usual type, filled with Trost mixture and having dimensions of 5 cm diam. $\times$ 90 cm. The resolving power of the counting circuit was $5 \cdot 10^{-6}$ s.

The form and the width of the energy band of the mesons counted depend on the conditions of the experiment. Taking into account the field strength, the geometry of the counter arrangement and the energy-loss of the μ-mesons in the pole-pieces, we have found that the energy band for zero thickness of rock above the apparatus has a width of

(2) E. BERETTA, I. FILOSOFO and B. SOMMACAL: *Nuovo Cimento*, **10**, 317 (1952).

0.4 GeV with a mean value of 1 GeV. With increasing thickness of rock, the particles counted by the apparatus belong to a growing band of energies at the surface due to the different routes taken through the rock which are permitted by the acceptance angle of the apparatus. We have calculated the width of the energy band for the greatest thickness of rock under which measurements were made. It was necessary to take into account the range-energy relationship for μ-mesons and the contours of the rock surface within the acceptance angle of the apparatus. As a result, the energy band at the surface which corresponds to the measurements recorded by the apparatus at this depth, has a width of 2.2 GeV and a mean value of 21 GeV. We have concluded that the width of the energy band, with respect to the mean energy of 21 GeV, is such that this measurement may still be considered differential even under the greatest thickness of rock which we have used.

At the same time measurements have been made without a field, using triple coincidences between counters lying in a straight line having the same acceptance angle as before. By means of these measurements, and taking into account the integral spectrum and the range-energy relationship for μ-mesons, it has been possible to calculate the equivalent thickness in g/cm² of air, and to define the energy band for the differential measurement of the excess for each thickness of rock.

The apparatus used for the preliminary measurements, which have already been published, was exactly the same as that described in the present communication, except that it only consisted of eight vertical channels of the type 2-2′-D described above. Very good agreement was obtained between results produced by the two sets of apparatus for measurements under the same depth of rock, and for this reason we have been able to sum together the number of mesons of each sign counted by both sets of apparatus for thicknesses of 1 400 and and of 4 050 g/cm².

Table I.

g/cm²	460	1 400	2 400	4 050	8 800
Mean energy of Band in Gev.	1	3.2	5.5	9.5	21
(P)	63 028	135 528	15 635	18 666	9 373
(N)	51 986	107 126	12 439	15 026	7 607
P/N	1.212 ±0.001	1.265 ±0.004	1.257 ±0.013	1.242 ±0.011	1.232 ±0.016
δ%	19.2 ±0.6	23.4 ±0.4	22.7 ±1.1	21.6 ±1.0	20.8 ±1.5

The results of the combined experiments are summarized in Table I and in Fig. 2.

As has been pointed out in the preceeding paper, which presented the first series of results, there is a very good agreement with the findings of the Manchester group [3] working at sea level with the magnetic spectrograph. This

[3] B. G. Owen and J. G. Wilson: *Proc. Phys. Soc.*, A **64**, 417 (1951).

seems to indicate that in both experiments we are only dealing with μ-mesons, and that what we observe underground are just the same particles that cross the earth's surface and which are slowed down by energy losses which are of electromagnetic origin only.

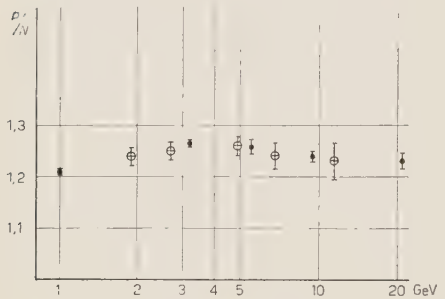

Fig. 2.

The new value at 21 GeV seems to confirm the peculiar fact, already pointed out, that the decrease of P/N ratio is a slowly varying function of the energy of the μ-mesons — at least in this energy range. This fact means that the mean multiplicity of production of π-mesons in the atmosphere is also a slowly varying function of the π's energy.

An investigation was also made into the possibility of a variation with local time of P/N ratio. For this purpose readings were taken every 3 hours and grouped together in order to detect a variation. No such variation has been found outside the statistical errors; so, if an effect does exist, the amplitude of the diurnal wave can hardly exceed 10% of the mean value.

We would like to thank the « Società Adriatica di Elettricità », and in particular the Director of the « Azienda Idroelettrica », Ing. M. MAINARDIS, for permission to work in their generating station. Our thanks must also go to Mr. A. CARESTIATO, Director of the Agordo group of generating stations, for the help and facilities made available to us at Agordo.

We are indebted to Prof. G. PUPPI for his advice and continuous encouragement during the long period of these experiments.

On the γ-Rays Associated with S-Particles

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B. Dequal

Istituto di Fisica dell'Università - Trieste

(ricevuto il 24 Settembre 1954)

When a charged particle is accelerated, there is a certain probability to emit a radiation γ-ray. Thus, in the decay of a positive π-meson at rest ($\pi^+ \to \mu^+ + \nu$), the presence of γ-ray emission has been deduced from the observed short range μ-meson.

The effect has been calculated by Primakoff [1] and by Eguchi [2], the latter calculation being relativistically invariant. The theoretical calculations agree, within the very poor statistics, with the experimental results of Fry [2,3], giving about one chance in 750 for a μ-meson to have less then 3.5 MeV. (The maximum energy of the μ-meson is 4.1 MeV).

We have considered the above effect for the decay of S-particles, assuming an S-particle of mass 980 m_e which decays into a μ-meson and a neutrino, i.e. the K_μ particle suggested by Gregory *et al.* [4]

$$K_\mu^+ \to \mu^+ + \nu$$

and have calculated the integral probability $P(E_\gamma)$ for γ-rays of energy greater than E_γ to be emitted in the above decay. Simply putting the numbers into the Eguchi formulae [2] integrated over μ-meson energy, the result given in Fig. 1 is found.

It can be seen that this probability is much larger than in the π-meson decay, mainly because of the much higher Q-value. There is about one chance in 108 to emit a γ-ray of energy greater than 5 MeV.

Again, using the Eguchi formulae, it is easy to show that the probability, $Q(\vartheta, E_\gamma)\,d\omega$, to emit a γ-ray of energy E_γ; at an angle ϑ with the μ-meson, into the solid angle $d\omega$ is simply

$$Q(\vartheta, E_\gamma) \alpha \sin^2 \vartheta \,,$$

(1) H. Primakoff: *Phys. Rev.*, **84**, 1255 (1951).
(2) T. Eguchi: *Phys. Rev.*, **85**, 934 (1952).
(3) W. F. Fry: *Nuovo Cimento*, **8**, 590 (1951); *Phys. Rev.*, **83**, 1268 (1951).
(4) B. Gregory, A. Lagarrigue, L. Leprince-Ringuet, F. Muller and Ch. Peyrou: *Nuovo Cimento*, **11**, 292 (1954).

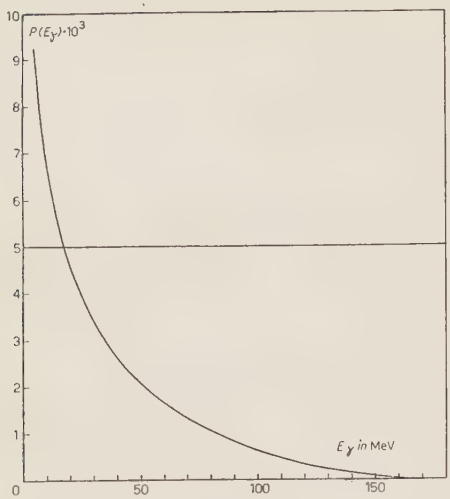

Fig. 1. – Integral Probability for γ-rays emission of energy greater than E_γ.

if E_γ is very much less than the μ-meson cinetic energy.

Thus, these γ-rays will tend to be emitted in the plane perpendicular to the line of flight of the μ-meson.

It seems evident that this effect is about a factor of 25 too small, and that the angular distribution is peaked too near 90° from the μ, to explain the γ-rays observed by BRIDGE *et al.* [5] associated with S-particle decays.

We are indebted to Professor N. DALLAPORTA for helpful discussions.

[5] H. BRIDGE, H. COURANT, B. DAYTON, H. C. DESTAEBLER, B. ROSSI, R. SAFFORD and D. WILLARD: *Nuovo Cimento*, **12**, 81 (1954).

IL NUOVO CIMENTO Vol. XII, N. 5 1° Novembre 1954

Mass Quantization in Non-Local Field Theory.

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(ricevuto 24 Settembre 1954)

In order to take account of the new unstable particles the author proposed in previous notes [1,2] a bilocal generalization of the Schrödinger-Gordon equation

$$(1)\qquad [[\psi, d_\mu]_-, d_\mu]_- + C^2[[\psi, d_\mu]_+, d_\mu]_+ = \varkappa^2 \psi$$

and of the Dirac equation

$$(2)\qquad \alpha_\mu[\psi, d_\mu]_- + iC\beta_\mu[\psi, d_\mu]_+ + \varkappa\psi = 0\,,$$

together with reciprocally invariant supplementary conditions

$$(3)\qquad [[\psi, a_\mu^+]_-, a_\mu^+]_- = [[\psi, a_\mu^-]_-, a_\mu^-]_- = b\psi$$

where d_μ is the displacement operator

$$(4)\qquad [x_\mu, d_\nu]_- = \delta_{\mu\nu}\,,$$

and $a_\mu^\pm$ are

$$(5)\qquad a_\mu^\pm = \frac{1}{\sqrt{2}}(\lambda^{-1}x_\mu \mp i\lambda d_\mu)\,.$$

b and C are dimensionless constants while λ is a constant with the dimension of length.

Going over to Yukawa variables [3] x_μ, r_ν the generalized Schrödinger-Gordon equation (1) becomes

$$(6)\qquad (p_\mu^2 + a^2 k_\mu^2 + \varkappa^2)\,\psi(x,r) = 0\,,$$

where

$$(7)\qquad p_\mu = -i\frac{\partial}{\partial x_\mu}\,,\quad k_\mu = -i\frac{\partial}{\partial r_\mu}\,,\quad a^2 = 4C^2\,.$$

Assuming natural units $\hbar = c = \lambda = 1$ the supplementary conditions (3) become

$$(8)\qquad (r_\mu - p_\mu)^2\psi = (r_\mu + p_\mu)^2\psi = b\psi\,.$$

The supplementary conditions (3) or (8) restrict the domain of variability of r_μ to polar angles φ, ϑ on a sphere with radius $\sqrt{b - p_\mu^2}$ in the system of rest of the particle.

The supplementary conditions may be taken explicitely into account by introducing into the equation (6) the reaction forces due to the constraints and

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(1) J. Rayski: *Nuovo Cimento*, **10**, 1729 (1953).

(2) J. Rayski: *Acta Phys. Pol.*, **13**, 77 (1954).

(3) H. Yukawa: *Phys. Rev.*, **77**, 219 (1950).

investigating a new function $\Phi(x_\mu, \varphi, \vartheta)$ of the independent variables. Thus, we get a modified equation

$$(9) \qquad \left(p_\mu^2 + \frac{a^2}{r_\nu^2} S^2 + \varkappa^2\right)\Phi = 0\,,$$

$$[r_\mu \cdot p_\mu = 0,\ r_\mu^2 = b - p_\mu^2]$$

where S^2 is (the relativistically invariant generalization of) the squared angular momentum operator with the eigenvalues $l\,(l+1)$.

In contradistinction to a previous supposition [1] it appears that this formalism has nothing to do with the isotopic spin but describes a family of particles with different spins (and masses). Thus, our formalism is closely connected with FIERZ's [4] formalism of higher spins with the only difference that, in our case, (i) all higher spin values come at once into play, and (ii) the masses of particles with spin $l=0, 1, \ldots$ form a spectrum with prescribed values

$$(10) \qquad M_l^2 = \frac{\varkappa^2 - b}{2} + \sqrt{\left(\frac{\varkappa^2 + b}{2}\right)^2 + a^2 l\,(l+1)}.$$

Assuming a pseudoscalar wave function ψ it is to be expected that (10) describes the mass spectrum of the pion family.

Putting (arbitrarily) $b=0$ we are left with two constants a and $\varkappa$ which may be determined from the two lowest values of masses (273 m_e and 965 m_e). Then, the higher mass eigenvalues follow from (10)

$$(11) \qquad M_2 = 1\,260\ m_e\,, \quad M_3 = 1\,490\ m_e, \text{ etc.}$$

These values seem to find experimental confirmation in the two observations from the Brookhaven Cosmotron [5] (1 280 m_e and 1 290 $m_e \pm 80\ m_e$) and in a statistics of tracks in the emulsion (1 485 $m_e \pm 50\ m_e$) [6].

Starting with the Dirac equation (2) we may obtain a mass spectrum for the nucleon family. A provisional determination of the constants a and $\varkappa$ from the two lowest mass eigenvalues (1840 m_e, 2 180 m_e) from the formula

$$(12) \qquad M_j^{\pm} = \pm\frac{\varkappa}{2} + \sqrt{\left(\frac{\varkappa}{2}\right)^2 + a\left(j + \frac{1}{2}\right)}\,, \qquad j \quad \tfrac{1}{2}, \tfrac{3}{2}, \ldots$$

(where we put again $b = 0$) yields the next masses 2 660 m_e and 3 000 m_e which seems to favour the decay schemes $\Omega^- \to \Lambda^0 + \pi^-$ and $Y^+ \to P^+ + \theta^0$ instead of the usually assumed $\Omega^- \to N + \pi^-$ and $Y^+ \to P^+ + \pi^0$.

However, the predicted numerical values are not very much reliable not only because of the arbitrary assumption $b=0$ but also on the following reason: most probably the constant $\varkappa$ is a field mass which is not a universal constant but depends upon the spin and the total mass of the particle in question.

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[4] M. FIERZ: *Helv. Phys. Acta*, **12**, 3 (1939); **23**, 412 (1950).

[5] A. THORNDIKE: *Int. Conf. Glasgow*, July 1954.

[6] D. H. PERKINS *Int. Conf. Glasgow*, July 1954.

The Harmonic mean Energy for Photon Absorption by Nuclei.

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(ricevuto il 1° Ottobre 1954)

In their paper on the nuclear photoeffect, Levinger and Bethe [1] derive an expression for the harmonic mean energy W_H for photon absorption as a function of the expectation value of the squared displacement of a nucleon in the nuclear ground state, $\langle r^2\rangle_{00}$:

$$(1) \qquad W_H = \frac{\int\sigma(E)\,\mathrm{d}E}{\int\sigma(E)\,\mathrm{d}E/E} = \frac{3\hbar^2}{2M}\,\frac{1+0.8x}{\langle r^2\rangle_{00}}\,.$$

Here $\sigma(E)$ is the nuclear absorption cross-section of photons with energy E, x is the fraction of neutron-proton exchange force, M the nucleon mass. Since the integrals which occur in equation (1) can be obtained from experiment, it is possible thereby to compute $\langle r^2\rangle_{00}$ and thus gain some insight on the nuclear structure.

We wish to report here computations which were carried out by utilizing extensive experimental results available from the work of Montalbetti, Katz and Goldemberg [2] and Goldemberg and Katz [3]. The experimental data refer to (γ, n) cross-sections for nuclei C, Ca, V, Ni, Zn, Br and Rb of Table I; and to $(\gamma, \mathrm{n}) + (\gamma, 2\mathrm{n}) + (\gamma, \mathrm{np})$ cross-sections for the nuclei Na, Al, P, S, Co, Cu, As, Mo, Nb, Ag, In, Sb, I, La, Ta, Au, Pb and Bi of the same table. For medium and heavy nuclei we may reasonably expect that such processes represent the major contribution to the cross-section for photon absorption.

In Table I are the harmonic mean energies W_H for a number of nuclei and the corresponding values of $\langle r^2\rangle_{00}$ as obtained from relation (1) for $x=0$ and $x=1$. The last column gives the values of $\langle r^2\rangle_{00}$ obtained from the plane-wave model of the nucleus: $\frac{3}{5}(r_0 A^{\frac{1}{3}})^2$, with $r_0 = 1.5\cdot 10^{-13}$ cm. The points in Fig. 1 reproduce the values of $\langle r^2\rangle_{00}$ for $x=0$, against A; the straight line parallel to the abcissae is the value of $\frac{3}{5}(r_0 A^{\frac{1}{3}})^2$ for the α-particle, $A=4$. The ascending curve represents the last column of the Table.

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(1) J. S. Levinger and H. A. Bethe: *Phys. Rev.*, **78**, 115 (1950).

(2) R. Montalbetti, L. Katz and J. Goldemberg: *Phys. Rev.*, **91**, 659 (1953).

(3) J. Goldemberg and L. Katz: *Can. Journ. Phys.*, **32**, 49 (1954).

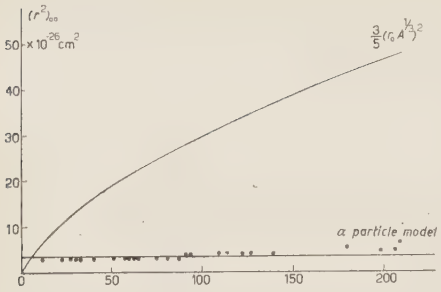

Fig. 1.

It is seen that the available experimental data strongly substantiate the model of a nucleus in which the nucleons are most of time clustered in α-particles as sub-units: $\langle r^2 \rangle_{00}$ does not increase with A and its value is of the order of magnitude expected for the mean square displacement of a nucleon inside an α-particle: $3.5 \cdot 10^{-26}$ cm^2. This model was already suggested by LEVINGER and BETHE for ^{76}Ge.

TABLE I.

Nucleus	A	W_B	$\langle r^2 \rangle_{00} \cdot 10^{-26}$ cm^2		$\langle r^2 \rangle_{00} = \frac{3}{5}(r_0 A^{\frac{1}{3}})^2 \cdot 10^{-26}$ cm^2
			$x=0$	$x=1$	
C	12	20 MeV	3.12	5.62	7.06
Na	23	18.7	3.34	6.02	10.9
Al	27	19.1	3.27	5.88	12.4
P	31	20.9	3.00	5.40	13.3
S	32	20.4	3.06	5.50	13.6
Ca	40	19.8	3.15	5.68	15.9
V	51	17.6	3.54	6.38	18.1
Co	59	17.2	3.63	6.54	20.5
Ni	58	18.4	3.40	6.12	20.3
Cu	63	17.2	3.64	6.55	21.4
Zn	64	18.0	3.46	6.23	21.6
Cu	65	17.3	3.61	6.50	21.8
As	75	19.2	3.25	5.85	24.0
Br	81	18.0	3.46	6.22	25.0
Rb	87	20.8	3.00	5.40	26.6
Mo	92	17.9	3.86	6.95	27.5
Nb	93	16.1	3.88	7.00	27.8
Ag	109	15.8	3.95	7.05	30.8
In	114	16.3	3.83	6.90	31.7
Sb	122	16.4	3.80	6.84	33.2
I	127	16.3	3.80	6.84	34.1
La	139	16.3	3.80	6.84	36.2
Ta	181	12.0	5.20	9.36	43.3
Au	198	15.2	4.10	7.38	46.0
Pb	207	13.8	4.54	8.17	47.2
Bi	209	10.0	6.25	11.20	47.5

Sull'urto nucleone-protone con produzione di mesoni.

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(ricevuto il 10 Ottobre 1954)

I risultati sperimentali preliminari comunicati in una precedente nota [1], inducono a concludere che nell'urto di elevata energia fra due nucleoni, la sezione d'urto per eventi con produzione di $N \geqslant 3$ particelle cariche è maggiore nel caso n-p della sezione d'urto nel caso p-p.

In assenza di una teoria della produzione multipla, applicheremo i principi generali di invarianza e di conservazione ed il principio di esclusione di Pauli generalizzato.

I due casi, urto n-p e p-p, che si presentano nella nostra esperienza, si distinguono per la coordinata di carica totale del sistema: nel primo caso ci si deve aspettare un numero dispari di particelle cariche, nel secondo caso un numero pari; però l'esperienza da noi eseguita con gli odoscopi non permette di conoscere la molteplicità effettiva dello sciame prodotto a causa del limitato potere risolutivo degli odoscopi e quindi le molteplicità pari e dispari registrate possono differire di qualche unità dalla molteplicità reale.

Indichiamo con $\sigma^N_{T=0}(\mathrm{np})$ e $\sigma^N_{T=1}(\mathrm{np})$ le sezioni totali per produzione di N pioni (carichi+neutri) con spin isotopico $T=0$ e $T=1$ rispettivamente, e indichiamo con $\sigma^N_{T=1}(\mathrm{pp})$ l'analoga sezione per l'urto p-p. Ovviamente se si considera l'urto di un neutrone incidente con un protone, il 50% degli stati iniziali corrisponde allo stato $T=0$, $T_z=0$, e il 50% allo stato $T=1$, $T_z=0$.

Assumendo la conservazione dello spin isotopico totale (trattandosi di eventi con interazione non elettromagnetica) e ricordando che non vi sono interferenze nelle sezioni d'urto totali fra gli stati di $T=1$ e $T=0$, possiamo scrivere sommando rispetto all'indice N:

$$\frac{\sigma(\mathrm{n,\,p})}{\sigma(\mathrm{p,\,p})} = \frac{\sum_{N\geq 3}\left[\frac{1}{2}\sigma^N_{T=1}(\mathrm{np}) + \frac{1}{2}\sigma^N_{T=0}(\mathrm{np})\right]}{\sum_{N\geq 3}\sigma^N_{T=1}(\mathrm{pp})} \geq \frac{1}{2}\,.$$

Infatti, assumendo la validità dell'ipotesi della «charge independence» si ha: $\sigma^N_{T=1}(\mathrm{np}) = \sigma^N_{T=1}(\mathrm{pp})$, per ogni N.

Dai nostri risultati sperimentali [2] riprodotti nella tabella I e aggiornati ri-

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(1) P. Colombino, S. Ferroni and G. Wataghin: *Nuovo Cimento*, **11**, 572 (1954).

(2) Questi risultati sono stati presentati al Congresso della S.I.F. a Parma, 6 Settembre 1954.

TABELLA 1.

	Paraffina (P) ore 2 369		Grafite (G) ore 2 369		Fondo (0) ore 1 260		$P-G$
	N	Frequenza	N	Frequenza	N	Frequenza	Frequenza
$SM_{(p)}$	1 068	$0{,}45 \pm 0{,}014$	1020	$0{,}43 \pm 0{,}013$	292	$0{,}22 \pm 0{,}012$	$0{,}02 \pm 0{,}02$
$SM_{(n)}$	1 469	$0{,}62 \pm 0{,}016$	1 118	$0{,}47 \pm 0{,}014$	299	$0{,}21 \pm 0{,}012$	$0{,}15 \pm 0{,}02$
SD	1 102	$0{,}46 \pm 0{,}014$	1 139	$0{,}48 \pm 0{,}014$	688	$0{,}5 \pm 0{,}02$	—

$$\frac{(P-G)_n}{(P-G)_p} = \frac{0{,}15 \pm 0{,}02}{0{,}02 \pm 0{,}02}$$

Effetto barometrico:

$SM_{(p)} \sim -10\%$
$SM_{(n)} \sim -10\%$
$SD \sim -20\%$

spetto a quelli della nota precedente, si deduce che il contributo degli stati iniziali con $T=0$ nell'urto n-p sembra dominante per gli eventi osservati, ossia per l'intervallo di energia dei nucleoni incidenti che è selezionato dalla apparecchiatura usata (energia $\gtrsim 10$ GeV).

L'esperienza è tuttora in corso.

Camera a scintillazione per la misura dell'emanazione contenuta nell'aria.

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(ricevuto il 19 Ottobre 954)

Le più recenti tecniche di rivelazione di particelle ionizzanti sono state utilizzate in questi ultimi anni per lo studio della radioattività α contenuta nell'aria [1-5]. Si è rivelato di particolare utilità l'impiego dei contatori a scintillazione che utilizzano la scintillazione delle particelle α in solfuro di zinco attivato con argento, (ZnS*).

Scopo di questa breve notizia è descrivere una camera a scintillazione che può essere usata, molto semplicemente e con buon risultato, per lo studio dell'emanazione radioattiva contenuta nell'aria.

La camera è illustrata nella fig. 1, ed è costituita da un tubo di ottone di sezione circolare col diametro di 50 mm e l'altezza di 25 mm e da due dischi che lo chiudono alle estremità, l'uno pure di ottone, l'altro di plexiglas. Nell'interno della camera, coassialmente alla parete laterale, sono disposti due tubi di celluloide sottile, pure a sezione circolare, rispettivamente di diametro 35 e 17 mm. Due rubinetti permettono di introdurre nella camera l'aria da esaminare.

Le superficie interne della camera, comprese le superficie dei tubi di celluloide ed esclusa la base di plexiglas, sono ricoperte da strati di ZnS*. Gli strati di ZnS* costituiscono oltre che una grande superficie di scintillazione, una buona superficie diffondente e favoriscono così la raccolta della luce sul fotocatodo del fotomoltiplicatore.

Questa disposizione geometrica è stata studiata per ottenere la massima efficienza possibile al conteggio delle particelle α.

Durante la misura la base di plexiglas è posta, con un opportuno contatto ottico, sul fotocatodo di un fotomoltiplicatore del tipo RCA 5819 o Dumond 6292.

Una convenzionale catena elettronica di amplificazione e registrazione permette il conteggio degli impulsi ottenuti dalle particelle α. La misura si fa riempiendo, la camera dell'aria da studiare e conteggiando direttamente le particelle α.

Nella disposizione della fig. 1 le particelle α emesse, per esempio, da un nucleo di radon, o da uno dei suoi discendenti RaA o RaC' o da eventuali altre con-

(1) G. Aliverti: *Ann. di Geofisica*, **1**, 372 (1948).
(2) A. Drigo: *Nuovo Cimento*, **7**, 501 (1950).
(3) P. E. Damon e H. I. Hyde: *Rev. Scient. Inst.*, **23**, 766 (1952).
(4) J. H. Harley: *Nucleonics*, **11**, N° 7, 12 (1953).
(5) G. Delibrias: *Journ. Phys. et le Rad.*, **15**, 78A, 80A (1954).

taminazioni radioattive α presenti nel volume della cameretta, prima di esaurire il loro percorso, colpiscono, con grande probabilità, lo strato di scintillatore e sono rivelate. Non vengono pertanto contate quelle particelle α che, essendo dirette verso la base di plexiglas, non incontrano lo scintillatore. Poichè il loro numero può ritenersi dell'ordine del 10% del totale, l'efficienza del conteggio è molto elevata e dell'ordine di 0,9. Per le note proprietà del ZnS* come scintillatore il dispositivo è praticamente insensibile ai raggi β e γ.

La sensibilità dello strumento nel conteggio di particelle α alle basse concentrazioni di radon è limitata dal rumore di fondo, numero d'impulsi ottenuti nella camera senza aria; tale numero è risultato dell'ordine di 10 impulsi all'ora, senza fare uso di particolari accorgimenti. Tenendo conto di tale fondo si può dire che concentrazioni di radon dell'ordine di 10^{-12} curie/litro possono essere misurate comodamente: con tale concentrazione si hanno infatti circa 18 impulsi all'ora. Volendo ottenere sensibilità maggiori la camera può essere usata in connessione con un impianto di concentrazione del radon, per esempio del tipo descritto da G. DELIBRIAS [5].

Viceversa nel caso delle concentrazioni elevate si può dire che la camera può servire a misurare con buona precisione fino a circa 10^{-6} curie/litro, corrispondenti pressapoco a 10^{8} impulsi all'ora, questo numero limite di impulsi è valutato tenendo conto della vita media del decadimento del ZnS* che è di qualche microsecondo; un tale numero può essere facilmente risolto e contato con un conveniente dispositivo elettronico.

Lo strumento presenta i noti svantaggi dovuti al fatto che l'emanazione viene messa in diretto contatto con il rivelatore. Si ha così che, una volta riempita la cameretta per una misura e rivuotata dell'aria, essa rimane contaminata e prima che sia di nuovo pronta occorre aspettare che i prodotti del radon a vita breve siano decaduti, ossia qualche ora. A questo inconveniente si può far fronte usando parecchie camerette alter-

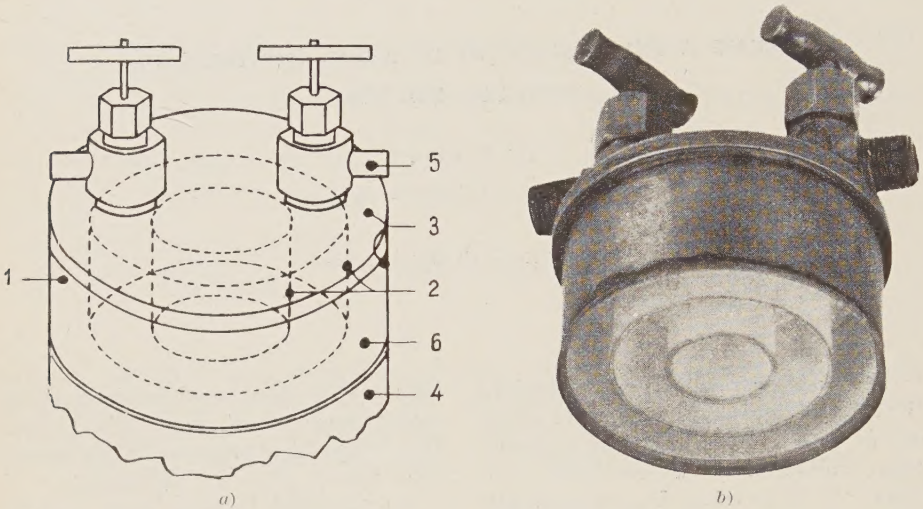

Fig. 1. – Camera a scintillazione: *a*) disposizione di conteggio: 1) camere a scintillazione; 2) cilindri di celluloide ricoperti di uno strato di ZnS*; 3) superfici ricoperte con uno strato di ZnS*; 4) fotomoltiplicatore; 5) rubinetti; 6) base di plexiglas – *b*) fotografia della camera a scintillazione.

nativamente, dato il loro basso costo. Poichè come prodotti a vita lunga del decadimento del radon si hanno RaD e polonio, c'è da aspettarsi che una debole attività α dovuta al polonio rimanga sempre presente nella cameretta e tenda ad aumentare con l'uso. Tale attività è però percentualmente del tutto trascurabile rispetto all'attività del Rn corrispondente, alla quale è succeduta.

L'attività residua verrebbe ad essere di disturbo come fondo solo nel caso in cui con la stessa cameretta si volessero misurare attività differenti fra loro di grossi fattori, dell'ordine di 10^5; in tal caso la contaminazione dovuta alla attività più grande potrebbe dare un numero d'impulsi maggiori di quelli dati dall'attività più piccola. Per ovviare a questo inconveniente conviene usare diverse camerette per misurare attività molto diverse fra loro.

Vantaggi dell'apparecchio sono viceversa la sua semplicità di costruzione e il suo basso costo, il suo semplice uso e in particolare il fatto che non occorre nessuna manipolazione dell'aria in misura.

Infine lo strumento fornisce rapidamente e senza necessità di taratura la concentrazione di radon contenuto nell'aria con una buona precisione. Difatti a questo scopo notiamo che l'efficienza del conteggio delle particelle α si può ritenere vicina all'unità e che, in condizioni di equilibrio, ad ogni nucleo di Rn corrispondono tre particelle α (Rn, RaA, RaC'). La concentrazione di radon può essere quindi dedotta direttamente dal conteggio.

Le curve di decadimento dell'attività, infine, possono essere seguite direttamente nella camera allo scopo di avere informazioni sullo stato di equilibrio radioattivo e di poter distinguere i vari emettitori α eventualmente presenti nell'aria.

Occorre infine dire che una camera basata sullo stesso principio ma con una geometria diversa e di minor efficienza è stata proposta da P. E. Damon e H. I. Hyde [3]

LIBRI RICEVUTI E RECENSIONI

C. A. MUSES – *An evaluation of relativity theory*. S. Weiser, Inc,. New York, 1953, pagine 48, $ 2,15.

Nel prossimo anno la relatività einsteniana compie 50 anni di vita. Parrebbe, con i tempi che corrono, che una teoria in grado di raggiungere tale venerabile età, debba aver già maturato il diritto ad essere riconosciuta come una solida acquisizione del pensiero scientifico ed è questa effettivamente l'opinione della maggior parte dei fisici. Si deve tuttavia anche ammettere che le verità scientifiche hanno storicamente il peso statistico dei loro sostenitori e poche teorie forse come la relatività hanno acceso nel passato le passioni della polemica.

Il volumetto che recensiamo vuol riportarci al calore di quelle battaglie ed è classificabile tra quelli che, pur accettando il « formulario » relativista, vogliono mettere in guardia la scienza contro la enormità degli errori « concettuali » nei quali essa è stata trascinata dall'opera di EINSTEIN. L'autore, un Ph.D. licenziato alla Columbia University, è in sostanza un « realista » che distingue tra « fenomeni in sè » e « distorsioni e apparenze » dovute a difficoltà metodologiche e metrologiche di osservazione. Fin qui va quasi tutto bene e anzi è un atteggiamento presso a poco di questo genere che ha fatto camminare molto spesso la scienza. Ma il guaio è che per l'autore i fenomeni « in sè » pare richiedano esclusivamente concetti e linguaggio di un certo ben determinato tipo (dobbiamo dirlo esplicitamente? naturalmente quelli codificati dalla geometria euclidea e dalla fisica classica). Come l'autore sia a conoscenza di questa preferenza dell'in sè non ce lo dice e noi quindi dobbiamo confessare che non abbiamo tratto alcun profitto dalla lettura e restiamo, anche concettualmente, seguaci della « so-called » teoria della relatività. Il libretto è, come si addice, naturalmente diviso in capitoli e dotato di appendici.

A. BORSELLINO

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